MA0301 ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2019

EXAM 2

Exercise 1 (Logic):	15 points
Exercise 2 (Relations):	15 points
Exercise 3 (Induction):	20 points
Exercise 4 (Functions):	15 points
Exercise 5 (Combinatorics):	15 points
Exercise 6 (Boolean algebra):	10 points
Exercise 7 (Finite state machines)	10 points
Total:	**100 points**

Note: In each of the exercises 1.1, 2.1, 4.1, 5.1, exactly one answer is correct.

Exercise 1. Logic (15 points)

(1) **(3 points)**

Which of the three statements is correct?

I) The negation of $\exists x \forall y (p(x,y) \to q(x,y))$ *is* $\forall x \exists y (p(x,y) \wedge \neg q(x,y))$.

II) The negation of $\exists x \forall y (p(x,y) \to q(x,y))$ *is* $\forall x \exists y (p(x,y) \vee \neg q(x,y))$.

III) The negation of $\exists x \forall y (p(x,y) \to q(x,y))$ *is* $\forall x \exists y (\neg p(x,y) \wedge q(x,y))$.

(2) **(5 points)** *Show that* $(p \wedge q) \wedge \neg(p \vee q)$ *is a contradiction by constructing its truth table.*

(3) **(7 points)** *Use the laws of logic only to show that* $\neg(p \vee q) \vee (\neg p \wedge q) \equiv \neg p$

Solution 1. 1) I is correct.

2)

p	q	$p \wedge q$	$p \vee q$	$\neg(p \vee q)$	$(p \wedge q) \wedge \neg(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

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3)

$$\begin{aligned}\neg(p \vee q) \vee (\neg p \wedge q) &\equiv (\neg p \wedge \neg q) \vee (\neg p \wedge q)\\ &\equiv \neg p \wedge (\neg q \vee q)\\ &\equiv \neg p \wedge T\\ &\equiv \neg p\end{aligned}$$

Exercise 2. Relations (15 points)

(1) **(3 points)** *Let $A = \{2, 4, 6, 12, 20\}$ be ordered by divisibility.*
Which of the three statements is correct?
I) The minimal and maximal elements are $2, 6$ respectively 20.

II) The minimal and maximal elements are 2 respectively 12, 20.

III) The minimal and maximal elements are $2, 4$ respectively 20.

(2) **(5 points)** *a) Draw the directed graph of the relation R on $A = \{2, 3, 4, 6, 9\}$ defined by*

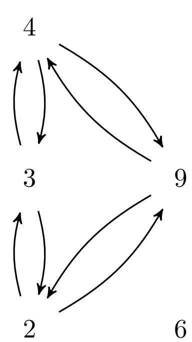
$$T = \{(2, 3), (2, 9), (3, 2), (3, 4), (4, 3), (4, 9), (9, 2), (9, 4)\}$$

(3) **(7 points)** *Let $A = \{1, 2, 3, 4, 5, 6\}$. Let R be the equivalence relation defined on A by:*

$$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$$

Determine the partition of A induced by R.

Solution 2. 1) II) is correct.
2)



3) $\{\{1, 5\}, \{2, 3, 6\}, \{4\}\}$ is the partition of A induced by R.

Exercise 3. Induction (20 points)

(1) **(8 points)** *Show that*

$$\sum_{k=1}^{n} (3k - 2) = \frac{n(3n - 1)}{2}$$

(2) **(12 points)** *First, show by induction that $n^2 \geq 2n + 1$ for $n > 2$. Use this result to determine by induction for which natural numbers we have that $2^n \geq n^2$.*

Solution 3. 1) Base step: $n = 1$: $3 - 2 = 1 = \frac{3-1}{2}$. Ind. hyp.: Assuming the statement is true for $k > 0$.

$$\sum_{k=1}^{n+1}(3k-2) = \frac{n(3n-1)}{2} + (3n+1) = \frac{n(3n-1)+2(3n+1)}{2} = \frac{(n+1)(3n+2)}{2} = \frac{(n+1)(3(n+1)-1)}{2}$$

2) Check that $3^2 = 9 \geq 6 + 1 = 7$. Hence, the formula holds for $n = 3$. Assume that it is true for $n > 2$, we check that $(n+1)^2 = n^2 + 2n + 1 \geq 2n + 1 + 2n + 1 = 2n + 2 + 2n \geq 2n + 2 + 1 = 2(n+1) + 1$. Thus the statements holds for $n + 1$.

Now, we move to the second statement. Check that for $n = 2$, we have $2^2 = 2^2$, but for $n = 3$, we have $2^3 = 8 < 3^2 = 9$, which violates the statement we want to prove. So, we look for $n = 4$ as base step, i.e., $16 = 4^2$ (for $n = 5$ we have $2^5 = 32 \geq 5^2 = 25$ and for $n = 6$ we have $2^6 = 64 \geq 36$). Assume that the statement holds for $n > 3$. For $2^{n+1} = 22^n \geq 2n^2 = n^2 + n^2 \geq n^2 + 2n + 1 = (n+1)^3$. Hence, by induction we have that the statement is correct.

Exercise 4. Functions (15 points)

(1) **(3 points)** *Which of the three statements is correct?*

I) The only solution of the equation $(n!)! = (2n)!$ is $n = 1$.

II) The equation $(n!)! = (2n)!$ holds for all $n \geq 4$.

III) $n = 0$ and $n = 3$ are all solutions of the equation $(n!)! = (2n)!$.

(2) **(5 points)** *Given $f : A \to B$ and $g : B \to C$. Show that if $g \circ f : A \to C$ is surjective, then g is surjective.*

(3) **(7 points)** *Let S be a set and let $P = \{A_1, \dots, A_k\}$ a partition of S. Since P is a partition of S we know that for each element $s \in S$ there exists a unique index $1 \leq j \leq k$ such that s is in $A_j \in P$. We define the map $f : S \to P$ by $f(s) = A_j$ if $s \in A_j$. Show that f is surjective.*

Solution 4. 1) III) is correct.

2) Let $a \in A$, then $g \circ f(a) = g(f(a))$ is in the image of g. Suppose that g is not surjective. Then the image of g is a proper subset of C and therefore the image of $g \circ f$ is a proper subset of C. The latter implies that $g \circ f$ is not surjective. Hence, we must have that g is surjective.

3) Choose an $A_i \in P$. As A_i is not empty there must exist an element $s \in S$ such that $s \in A_i$. Therefore $f(s) = A_i$. Hence, f is surjective.

Exercise 5. Combinatorics (15 points)

(1) **(3 points)** *Let $p_1, p_2, \dots, p_{12}$ be twelve given points in the plane. Suppose that no three of the points are on the same line. How many triangles, having vertices among the points $p_1, p_2, \dots, p_{12}$ and one of which is p_1, are there?*

Which one of the three statements is correct?

I) 220.

II) 55.

III) 165.

(2) **(4 points)** *Find the number of distinct permutations that can be formed from all the letters in the word RADAR.*

(3) **(8 points)** *We define the Narayana numbers:*

$$\begin{aligned} N(0,0) &= 1 \\ N(n,0) &= 0, \;\; n > 0 \\ N(n,k) &= \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \\ &= \frac{1}{n}\frac{n!}{k!(n-k)!}\frac{n!}{(k-1)!(n-k+1)!}, \;\; n \geq k \geq 1. \end{aligned}$$

Show that

$$N(n, n+1-k) = N(n,k).$$

Solution 5. 1) II

2) There are $30 = 5!/(2!2!)$, because there are five letters of which two are R and two are A.

3)

$$\begin{aligned} N(n, n+1-k) &= \frac{1}{n}\binom{n}{n+1-k}\binom{n}{n+1-k-1} \\ &= \frac{1}{n}\binom{n}{n+1-k}\binom{n}{n-k} \\ &= \frac{1}{n}\binom{n}{k-1}\binom{n}{n-k} \\ &= \frac{1}{n}\binom{n}{n-k}\binom{n}{k-1} \\ &= \frac{1}{n}\binom{n}{k}\binom{n}{k-1} = N(n,k) \end{aligned}$$

Exercise 6. Boolean algebra (10 points)

(1) **(5 points)** *Let B be a Boolean algebra. Show, using only the axioms of Boolean algebra (i.e., do not use any tables or alike), that for any elements $a, b \in B$ we have i) $\overline{a+b} = \bar{a} \cdot \bar{b}$ and ii) $\overline{a \cdot b} = \bar{a} + \bar{b}$.*

(2) **(5 points)** *Let B be a Boolean algebra and let a, b be any elements in B. Show, using only the axioms of Boolean algebra (i.e., do not use any tables or alike), that $\bar{a} + b = 1$ is equivalent to $a \cdot \bar{b} = 0$.*

Solution 6. 1) i) We must show that $(a+b) + \bar{a} \cdot \bar{b} = 1$ and $(a+b) \cdot (\bar{a} \cdot \bar{b}) = 0$. Uniqueness of complement then says that $\overline{a+b} = \bar{a} \cdot \bar{b}$.

$(a+b) + \bar{a} \cdot \bar{b} = b + (a + \bar{a}) \cdot (a + \bar{b}) = b + 1 \cdot (a + \bar{b}) = b + a + \bar{b} = 1 + a = 1.$

$(a+b) \cdot (\bar{a} \cdot \bar{b}) = ((a+b) \cdot \bar{a}) \cdot \bar{b} = ((a \cdot \bar{a}) + (b \cdot \bar{a})) \cdot \bar{b} = (0 + (b \cdot \bar{a})) \cdot \bar{b} = (b \cdot \bar{a}) \cdot \bar{b} = (b \cdot \bar{b}) \cdot \bar{a} = 0 \cdot \bar{a} = 0$

ii) $\overline{a \cdot b} = \bar{a} + \bar{b}$ follows similarly (or by duality).

2) Start by assuming that $\bar{a} + b = 1$ holds. Then $0 = \bar{1} = \overline{(\bar{a} + b)} = \bar{\bar{a}} \cdot \bar{b} = a \cdot \bar{b}$.
Now we assume that $0 = a \cdot \bar{b}$ is true. Then $1 = \bar{0} = \overline{(a \cdot \bar{b})} = \bar{a} + \bar{\bar{b}} = \bar{a} \cdot b$.

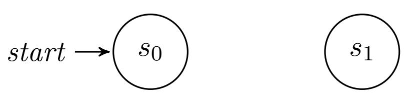
Exercise 7. Finite state machines (10 points)

(1) **(4 points)** *Draw the state diagram of the finite state machine N corresponding to the transition table*

N	ν		ω	
	0	1	0	1
s_0	s_0	s_1	0	0
s_1	s_1	s_2	0	0
s_2	s_2	s_3	0	0
s_3	s_3	s_4	0	0
s_4	s_4	s_5	0	0
s_5	s_5	s_0	0	1

What is the output corresponding to the input sequence 0110111011*?*

(2) **(6 points)** *Given two states s_0, s_1, complete the following diagram by adding arrows:*



so that it becomes a state diagram of the finite state machine M, which is supposed to recognise (with an output 1*) every* 0 *appearing in an input string x that is preceded by another* 0*.*

Solution 7. 1)

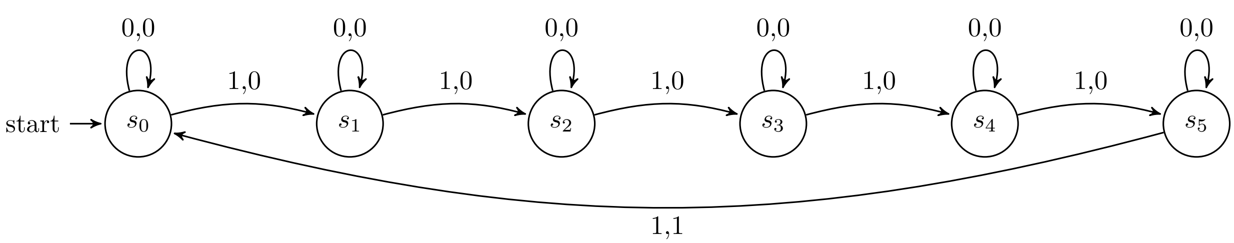


FIGURE 1. The finite state machine N.

Output: 0 0 0 0 0 0 0 0 1 0

2)

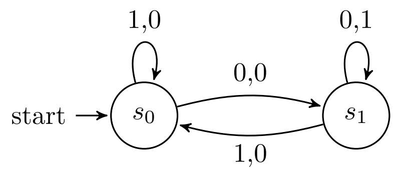


FIGURE 2. The finite state machine M.