MA0301 ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2018

Exam 2

Exercise 1 (Logic):	10 points
Exercise 2 (Relations):	20 points
Exercise 3 (Induction):	30 points
Exercise 4 (Functions):	15 points
Exercise 5 (Fibonacci numbers):	10 points
Exercise 6 (Graphs):	15 points

Total: 100 points

Exercise 1. Logic (10 points)

(1) **(4 points)** *Give the truth table for the statement*

$$(p \wedge (\neg q)) \longrightarrow r$$

(2) **(6 points)** *Use the laws of logic to show that* $((\neg p)\vee(\neg q))\wedge(F_0\vee p)\wedge p$ *is logically equivalent to* $p \wedge (\neg q)$. *Recall that* F_0 *denotes any contradiction.*

Solution 1. 1) Truth table for the statement $(p \wedge (\neg q)) \longrightarrow r$:

p	q	r	$\neg q$	$p \wedge (\neg q)$	$(p \wedge (\neg q)) \to r)$
T	T	T	F	F	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T

2) Show that $\big(((\neg p) \vee (\neg q)\big) \wedge (F_0 \vee p) \wedge p\big) \Leftrightarrow (p \wedge (\neg q))$

$$\begin{aligned}
\big((\neg p) \vee (\neg q)\big) \wedge (F_0 \vee p) \wedge p &\Leftrightarrow \big((\neg p) \vee (\neg q)\big) \wedge (p \wedge p)\\
&\Leftrightarrow \big((\neg p) \vee (\neg q)\big) \wedge p\\
&\Leftrightarrow (\neg p \wedge p) \vee (\neg q \wedge p)\\
&\Leftrightarrow F_0 \vee (\neg q \wedge p)\\
&\Leftrightarrow (\neg q \wedge p)\\
&\Leftrightarrow p \wedge (\neg q)
\end{aligned}$$

Date: August 12, 2018.

Exercise 2. Relations (20 points)

(1) **(3 points)** *One of the following definitions is correct. Which one is it?*

I) A relation R on a set A is called a partial order, or a partial ordering relation, if R is reflexive, anti-symmetric, and transitive.

II) A relation R on a set A is called a partial order, or a partial ordering relation, if R is anti-reflexive, anti-symmetric, and transitive.

III) A relation R on a set A is called a partial order, or a partial ordering relation, if R is reflexive, symmetric, and transitive.

(2) **(7 points)** *Define the relation R on the integers $\mathbb{Z}$ in the following way: for all $a, b \in \mathbb{Z}$, we have aRb if and only if $a + b$ is an even number. Prove or disprove (by a counterexample) that R is a partial ordering relation.*

(3) **(10 points)** *Define the relation R on the set $S := \{0, 1, 2, 3\}$:*

$$R := \{(0, 0), (0, 2), (1, 0), (1, 3), (2, 2), (3, 0), (3, 1)\}.$$

Draw the directed graph for the relation R. Prove or disprove that this relation is anti-symmetric.

Solution 2. 1) I is the right definition.

2) R is not a partial ordering relation on $\mathbb{Z}$. As a counterexample we consider $3 + 5 = 8$, which is even, and therefore $3R5$ and $5R3$ but we do not have $5 = 3$, i.e., we have $5 \neq 3$. This implies that R is not anti-symmetric.

3) R is not anti-symmetric, because $3R1$ and $1R3$ but $1 \neq 3$. See Figure 1.

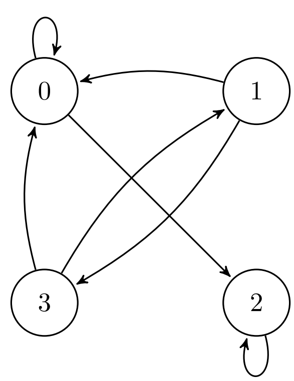

FIGURE 1. Exercise 2, solution 3)

Exercise 3. Induction (30 points)

(1) **(10 points)** *Use induction to show that for $n > 0$:*

$$\sum_{k=1}^{n} k(k!) = (n+1)! - 1$$

(2) **(20 points)** *Recall the definition of the Lucas numbers, i.e., $L_0 = 2$, $L_1 = 1$, and $L_k = L_{k-1} + L_{k-2}$ for $k > 1$.*

a) **(10 points)** *Use induction to show that for $n \geq 0$:*

$$\sum_{r=0}^{n} L_{2r} = L_{2n+1} + 1.$$

b) **(10 points)** *Use induction to show that for $n \geq 0$, L_{3n} is an even number.*

Solution 3. 1) Let $n = 1$: $\sum_{k=1}^{1} k(k!) = 1! = (1+1)! - 1$. We assume that the statement holds for $n = k$, i.e., $\sum_{j=1}^{k} j(j!) = (k+1)! - 1$. We want to show that it holds for $n = k+1$.

$$\sum_{j=1}^{k+1} j(j!) = \sum_{j=1}^{k} j(j!) + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)! = (k+1)!((k+1)+1) - 1 = (k+2)! - 1.$$

2)a) Let $n = 0$: $\sum_{r=0}^{0} L_{2r} = L_0 = 2 = L_1 + 1$. We assume that the statement holds for $n = k$, i.e., $\sum_{j=0}^{k} L_{2j} = L_{2k+1} + 1$. We want to show that it holds for $n = k+1$.

$$\sum_{j=0}^{k+1} L_{2j} = \sum_{j=0}^{k} L_{2j} + L_{2k+2} = L_{2k+1} + 1 + L_{2k+2} = L_{2k+3} + 1 = L_{2(k+1)+1} + 1.$$

2)b) Let $n = 0$: $L_0 = 2$. We assume that the statement holds for $n = k$, i.e., L_{3k} is even. We want to show that it holds for $n = k+1$.

$$L_{3k+3} = L_{3k+2} + L_{3k+1} = L_{3k+1} + L_{3k} + L_{3k+1} = 2L_{3k+1} + L_{3k}.$$

This is an even number since L_{3k} is even.

Exercise 4. Functions (15 points)

(1) **(3 points)** *One of the following definitions is correct. Which one is it?*

I) A function $f\colon A \to B$ is called surjective (onto) if $f(A) \subset B$.

II) A function $f\colon A \to B$ is called surjective (onto) if $f(A) = B$.

III) A function $f\colon A \to B$ is called surjective (onto) if for all $a_1, a_2 \in A$, whenever $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

(2) **(5 points)** *The function $f\colon \mathbb{Z} \to \mathbb{Z}$ is defined for all integers by $f(n) := 4n - 1$. Prove or disprove (by a counterexample) that f is surjective.*

(3) **(7 points)** *Let $A := \{1, 2, 3, 4\}$ and $B := \{5, 6, 7\}$. Define a function $f\colon A \to B$ that is surjective but not injective.*

Solution 4. 1) II is the correct definition of a surjective function.

2) Note that $0 \notin f(\mathbb{Z})$, which implies that $f(\mathbb{Z}) \neq \mathbb{Z}$.

3) Define $f(1) = 5$, $f(2) = 6$, $f(3) = 7$, and $f(4) = 7$.

Exercise 5. Fibonacci numbers (10 points)

(1) **(3 points)** *One of the three statements is correct. Which one is it?*

I) The Fibonacci numbers may be defined recursively by: $F_0 = 2$, $F_1 = 1$ *and* $F_n = F_{n-1} + F_{n-2}$ *for positive integers* $n > 1$.

II) The Fibonacci numbers may be defined recursively by: $F_0 = 0$, $F_1 = 1$ *and* $F_n = F_{n-1} + F_{n-2}$ *for positive integers* $n > 1$.

III) The Fibonacci numbers may be defined recursively by: $F_0 = 0$, $F_1 = 1$ *and* $F_n = F_{n-1} - F_{n-2}$ *for positive integers* $n > 1$.

(2) **(7 points)** *The Fibonacci numbers* $F_2, \dots, F_7$ *are:* $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$. *Prove that for any fixed* $n \geq 0$:

$$\sum_{r=0}^{5} F_{n+r} = 4F_{n+4}.$$

Hint: do not use the method of induction.

Solution 5. 1) The correct statement is II.

2) For any fixed $n \geq 0$ we want to show that $\sum_{r=0}^{5} F_{n+r} = 4F_{n+4}$.

$$\begin{aligned}
\sum_{r=0}^{5} F_{n+r} &= F_n + F_{n+1} + F_{n+2} + F_{n+3} + F_{n+4} + F_{n+5} \\
&= (F_n + F_{n+1}) + F_{n+2} + F_{n+3} + F_{n+4} + (F_{n+3} + F_{n+4}) \\
&= 2F_{n+2} + 2F_{n+3} + 2F_{n+4} \\
&= 2(F_{n+2} + F_{n+3}) + 2F_{n+4} \\
&= 2F_{n+4} + 2F_{n+4} \\
&= 4F_{n+4}.
\end{aligned}$$

Exercise 6. Graphs (15 points)

(1) **(3 points)** *One of the three statements is correct. Which one is it?*

I) Let $G = (V, E)$ be an undirected graph (or multigraph). Then $\sum_{v \in V} \deg(v) = |E|$.

II) Let $G = (V, E)$ be an undirected graph (or multigraph). Then $\sum_{v \in V} \deg(v) = 2|E|$.

III) Let $G = (V, E)$ be an undirected graph (or multigraph). Then $\sum_{v \in V} \deg(v) = 3|E|$.

(2) **(3 points)** *Can you find an undirected graph with 4 vertices of degrees 1, 2, 3, and 3?*

(3) **(3 points)** *An undirected graph has vertices of degrees 2, 4, 5, and 11. How many edges does the graph have?*

(4) **(6 points)** *Recall that the complete graph K_N is an undirected graph with N vertices and an edge between every two vertices. Show that K_N has $N(N-1)/2$ edges.*

Solution 6. 1) II is correct.

2) No, such a graph $G = (V, E)$ can not exist, because $\sum_{v \in V} \deg(v) = 1 + 2 + 3 + 3 = 9$, which is an odd number.

3) This graph has $\frac{1}{2} \sum_{v \in V} \deg(v) = \frac{1}{2}(2 + 4 + 5 + 11) = 11 = |E|$ edges.

4) Each of the N vertices of K_N has degree $N - 1$. Therefore, $\sum_{v \in V_{K_N}} \deg(v) = N(N-1) = 2|E|$. This implies that $|E| = N(N-1)/2$.