MA0301 ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2017

Exam 1

Exercise 1: 15 points
Exercise 2: 10 points
Exercise 3: 15 points
Exercise 4: 15 points
Exercise 5: 20 points
Exercise 6: 15 points
Exercise 7: 10 points

Total: 100 points

Exercise 1. Sets (15 points) *Use only the laws of set theory to proof the following statements for arbitrary sets A, B, C.*

(1) **(7 points)**

$$\text{If} \quad (A \cup B) \subseteq (A \cap B) \quad \text{then} \quad A = B.$$

(2) **(4 points)**

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

(3) **(4 points)**

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Solution 1. (1) Let's assume that $(A \cup B) \subseteq (A \cap B)$. We want to show that $A = B$, i.e., $A \subseteq B$ and $B \subseteq A$. Let's assume that $x \in A$

$$x \in A \Rightarrow x \in A \cup B \Rightarrow x \in A \cap B \Rightarrow x \in B.$$

Now we assume that $x \in B$

$$x \in B \Rightarrow x \in A \cup B \Rightarrow x \in A \cap B \Rightarrow x \in A.$$

This implies $A = B$.

(2)

$$x \in \overline{A \cap B} \Leftrightarrow x \notin (A \cap B) \Leftrightarrow (x \notin A) \text{ or } (x \notin B) \Leftrightarrow (x \in \overline{A}) \text{ or } (x \in \overline{B}) \Leftrightarrow x \in \overline{A} \cup \overline{B}.$$

(3)

$$\begin{aligned} x \in A \cap (B \cup C) &\Leftrightarrow (x \in A) \text{ and } (x \in (B \cup C)) \\ &\Leftrightarrow (x \in A) \text{ and } ((x \in B) \text{ or } (x \in C)) \\ &\Leftrightarrow ((x \in A) \text{ and } (x \in B)) \text{ or } ((x \in A) \text{ and } (x \in C)) \\ &\Leftrightarrow ((x \in (A \cap B)) \text{ or } ((x \in (A \cap C)) \Leftrightarrow (A \cap B) \cup (A \cap C) \end{aligned}$$

Date: August 28, 2017.

Exercise 2. Logic (10 points)

(1) **(6 points)** *Use the laws of logic to simplify:*

$$\big(p \vee (p \wedge q) \vee (p \wedge q \wedge \neg r)\big) \wedge \big((p \wedge r \wedge t) \vee t\big)$$

(2) **(4 points)** *Use a truth table to show that:*

$$\big((a \wedge b) \longrightarrow c\big) \Leftrightarrow \big((a \longrightarrow c) \vee (b \longrightarrow c)\big)$$

Solution 2. (1) We want to simplify $\big(p \vee (p \wedge q) \vee (p \wedge q \wedge \neg r)\big) \wedge \big((p \wedge r \wedge t) \vee t\big)$.

$$\begin{aligned}
&\big(p \vee (p \wedge q) \vee (p \wedge q \wedge \neg r)\big) \wedge \big((p \wedge r \wedge t) \vee t\big)\\
&\Leftrightarrow \big(p \vee ((p \wedge q) \wedge T) \vee ((p \wedge q) \wedge \neg r)\big) \wedge \big(((p \wedge r) \wedge t) \vee (T \wedge t)\big)\\
&\Leftrightarrow \big(p \vee ((p \wedge q) \wedge (T \vee \neg r))\big) \wedge \big(((p \wedge r) \vee T) \wedge t\big)\\
&\Leftrightarrow \big(p \vee ((p \wedge q) \wedge T\big) \wedge \big(T \wedge t\big)\\
&\Leftrightarrow \big(p \vee (p \wedge q)\big) \wedge t\\
&\Leftrightarrow p \wedge t
\end{aligned}$$

(2) The truth table for $\big((a \wedge b) \longrightarrow c\big) \Leftrightarrow \big((a \longrightarrow c) \vee (b \longrightarrow c)\big)$ is

a	b	c	$a \wedge b$	$a \longrightarrow c$	$b \longrightarrow c$	x	y	z
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	F	T	T	T	T	T
F	T	F	F	T	F	T	T	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

where $x := (a \wedge b) \longrightarrow c$ $\quad y := (a \longrightarrow c) \vee (b \longrightarrow c)$ $\quad z := ((a \wedge b) \longrightarrow c) \longleftrightarrow ((a \longrightarrow c) \vee (b \longrightarrow c))$.

Exercise 3. Equivalence relation (15 points)

(1) **(3 points)** *Write down the definition of an equivalence relation.*

(2) **(2 points)** *Write down the definition of an equivalence class.*

(3) **(10 points)** *Let $A := \{1, 2, 3\}$. Determine whether the following relations on A are equivalence relations. Give an argument in each case. If an equivalence relation is given determine the equivalence classes.*

- *i)* (5 points) $R_1 := \{(1,1), (2,2), (3,3), (1,2), (2,1), (3,1), (1,3)\}$
- *ii)* (5 points) $R_2 := \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2), (1,3), (3,1)\}$

Solution 3. (1) An equivalence relation R on the set $A(\neq \emptyset)$ is reflexive, symmetric, and transitive.

(2) Let R be an equivalence relation on the set $A(\neq \emptyset)$. For each $a \in A$, let $[a]$ denote the set of elements in A to which a is related through R. The set $[a]$ is called equivalence class of a in A.

(3) No, it is not transitive.

(4) Yes. Its only equivalence class is $\{1, 2, 3\}$.

Exercise 4. Functions (15 points)

(1) **(2 points)** *Give the definition of a surjective (onto) function.*

(2) **(3 points)** *Give the definition of a injective (one-to-one) function.*

(3) **(5 points)** *Let $g : A \to B$ and $f : B \to C$ be two functions. Prove that if g and f are both injective, then $f \circ g : A \to C$ is injective.*

(4) **(5 points)** *Define the function $f : \mathbb{N} \to \mathbb{N}$ by $f(n) := 2n$. Show that f is injective and that f is not surjective.*

Solution 4. (1) A function $f : A \to B$ is called surjective (or onto) if for every $b \in B$ there exists an $a \in A$ such that $f(a) = b$, i.e., f is onto if $f(A) = B$.

(2) A function $f : A \to B$ is called injective (or one-to-one), if each element of B appears at most once as the image of an element A, i.e., if all elements of A have different images: $x, y \in A$, $f(x) = f(y) \Rightarrow x = y$.

(3) Suppose $f \circ g(x) = f \circ g(y)$. Then $f(g(x)) = f(g(y))$ and $g(x) = g(y)$ since f is injective. Furthermore, $x = y$ since g is injective.

(4) It is not surjective since odd natural numbers are not in the image of f. It is injective by direct check following the definition.

Exercise 5. Induction (20 points)

(1) **(5 points)** *Show by induction that for all natural numbers*

$$\sum_{k=1}^{n} k(k+2)(k+4) = \frac{1}{4} n(n+1)(n+4)(n+5).$$

(2) **(7 points)** *Prove by induction that for all positive integers*

$$2 + 6 + 10 + \cdots + (4n - 2) = 2n^2.$$

(3) **(8 points)** *Show by induction that $n^3 - n$ is divisible by 3 for any positive integer n. (Recall that a positive integer m is divisible by 3 provided that there exists a positive integer t so that $m = 3t$).*

Solution 5. (1) Base step: for $n = 1$ we find $1 \times 3 \times 5 = 15$ on the lefthand side, and $1/4 \times 1 \times 2 \times 5 \times 6 = 60/4 = 15$, i.e., the statement holds for $n = 1$. Induction step: assume the statement holds up to k,

$\sum_{i=1}^{k} i(i+2)(i+4) = \frac{1}{4}k(k+1)(k+4)(k+5)$. For $n = k+1$ the lefthand side is then

$$\frac{1}{4}k(k+1)(k+4)(k+5) + 4(k+1)(k+1+2)(k+1+4)/4$$
$$= (k+1)(k+1+4)(k+2)(k+6)/4$$
$$= (k+1)(k+1+1)(k+1+4)(k+1+5)/4,$$

which gives the righthand side.

(2) Base step: for $n = 1$ we find $2 = 2 \times 1^2 = 2$, i.e., the statement holds for $n = 1$. Induction step: assume the statement holds up to k

$$2 + 6 + 10 + \cdots + (4k-2) = 2k^2.$$

For $n = k+1$ we find

$$2 + 6 + 10 + \cdots + (4k-2) + (4k+2) = 2k^2 + (4k+2)$$
$$= 2k^2 + 4k + 2 = 2(k^2 + 2k + 1) = 2(k+1)^2,$$

which is what was to prove.

(3) Base step: for $n = 1$ we find $1 - 1 = 0$, which is divisible by 3, i.e., $0 = 3 \times 0$, i.e., the statement holds for $n = 1$. Induction step: assume the statement holds up to k. Consider $(k+1)^3 - (k+1)$

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1) = k^3 - k + 3(k^2 + k) = 3m + 3(k^2 + k) = 3m',$$

where $m' := m + (k^2 + k)$.

Exercise 6. Finite state automata (15 points)

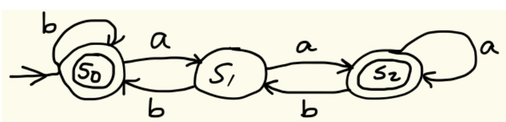
(1) **(10 points)** *Draw the state diagram $D(M)$ of the automaton M with states $S := \{s_0, s_1, s_2\}$, accepting states $Y := \{s_0, s_2\}$, input alphabet $I := \{a, b\}$, described in the following state table $T(M)$:*

	ν	
	a	b
s_0	s_1	s_0
s_1	s_2	s_0
s_2	s_2	s_1

(2) **(5 points)** *Which of the following input words are accepted by M and which are not accepted by M?*

1) bbaab
2) abbab
3) aabbb
4) babaab
5) aaabbb

Solution 6. (1)



(2) 1) not accepted; 2) accepted; 3) accepted; 4) not accepted; 5) accepted

Exercise 7. Graphs (10 points)
(10 points) *Let G be an arbitrary finite connected planar graph with at least three vertices. Show that G contains at least one vertex of degree equal or smaller than five.*

Solution 7. Recall that the sum of degrees of vertices equals $2|E|$, where E is the set of edges of the graph G; V is the set of vertices of G. We assume that there are at least three vertices. Recall that $2|E| \leq 6|V| - 12$. If every vertex has degree bigger than 5, then the sum of degrees of vertices is greater or equal than $6|V|$. Hence $2|E| \geq 6|V|$, which contradicts $2|E| \leq 6|V| - 12$. Hence, there must be at least one vertex of degree five or smaller.