MA0301
ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2023

Exercise Set 7 - Solutions

Deadline: Monday 13.03.2022, 23:59

The relevant sections of Lewis–Zax is chapters 4, 8, and 13.

Exercise 1. *Use induction to prove the following.*

a) *Show that for all natural numbers $n \geq 5$, there are natural numbers a and b such that $n = 2a + 5b$.*

Hint. *Use multiple base cases.*

b) *Show that for all natural numbers $n < 5$ there are integers a and b such that $n = 2a + 5b$.*

c) *Use a) and b) to show that every integer can be written as a sum $2a + 5b$ where a and b are integers.*

d) *(Optional) Describe a more direct strategy to prove c).*

Solution. a) Let the base cases be $n = 5$ and $n = 6$. If $n = 5$, then $n = 2 \cdot 0 + 5 \cdot 1$. If $n = 6$, then $n = 2 \cdot 3 + 5 \cdot 0$. Now, our inductive hypothesis can be defined as follows. If $5 \leq m \leq n$, then m can be written as a sum $m = 2a + 5b$.

To prove the inductive step, we must show that $n+1$ can be written as a sum $2a+5b$ when $n \geq 6$, given the inductive hypothesis. Now, in this case, we know that $n+1-2 = n-1 \geq 5$, so by the inductive hypothesis, we can write $n - 1 = 2a + 5b$ for natural numbers a and b. And since $n + 1 = (n - 1) + 2$, $n + 1$ can be written as the sum $n + 1 = 2 \cdot (a + 1) + 5b$.

b) We prove this by exhausting the cases. We have
- $0 = 2 \cdot 0 + 5 \cdot 0$
- $1 = 2 \cdot (-2) + 5 \cdot 1$
- $2 = 2 \cdot 1 + 5 \cdot 0$
- $3 = 2 \cdot (-1) + 5 \cdot 1$
- $4 = 2 \cdot 2 + 5 \cdot 0$

c) Together, *a*) and *b*) show that every natural number may be written as a sum $2a + 5b$ where a and b are integers. If n is negative, then we know $-n$ may be written as a sum $2a + 5b$, and in this case the following is a solution:

$$n = -(-n) = -(2a + 5b) = 2(-a) + 5(-b).$$

d) Since 1 can be written as the sum $1 = 2 \cdot (-2) + 5 \cdot 1$, we can write any integer n as

$$n = n \cdot 1 = n \cdot (2 \cdot (-2) + 5 \cdot 1) = 2 \cdot (-2n) + 5 \cdot n.$$

□

Exercise 2 (The fundamental theorem of arithmetic)**.** *Use strong induction to show that every natural number greater than 1 can be written as a product of primes.*[1]

[1]The primes themselves are considered "unary products" of primes.

Hint. *Use the inductive hypothesis that every number n satisfying $2 \le n \le m$ can be written as a product of primes $n = p_1 p_2 \cdots p_r$ for some positive integer r.*

Solution. The base case is $n = 2$ which is prime.
To show the inductive step, we must prove that $n+1$ can be written as a product of primes; that is, $n+1 = p_1 p_2 ... p_r$ where the p_i are prime.
There are two cases to consider. Either $n+1$ is prime, in which case we are done, or $n+1$ is a composite number. If $n+1$ is composite, it can be written as a product of smaller factors $n+1 = ab$ such that $2 \le a \le n$ and $2 \le b \le n$. Now, by the inductive hypothesis, a and b may be written as products of primes

$$a = q_1 q_2 \cdots q_s \quad \text{and} \quad b = q'_1 q'_2 \cdots q'_{s'}.$$

So $n+1$ may also be written as a product of primes, namely

$$n+1 = a \cdot b = q_1 \cdots q_s \cdot q'_1 \cdots q'_{s'}.$$

hence by strong induction, we have shown that every natural number greater than 1 can be written as a product of primes. □

Exercise 3. *Lewis–Zax: Exercise 8.1.*
Recall that strings are inductively defined as either the empty string λ (greek letter lambda), or a pair $\langle a, s\rangle$ where a is a symbol from some alphabet Σ (capital greek letter sigma) and s is another string.

Solution. We define $\#_a(s)$ inductively.
Base case: $\#_a(\lambda) = 0$.
Constructor case. Given the string $\langle x, s\rangle$, then $\#_a(\langle x, s\rangle)$ is defined as

- If $x = a$ then $\#_a(\langle x, s\rangle) := \#_a(s) + 1$.
- Otherwise $\#_a(\langle x, s\rangle) := \#_a(s)$.

We want to prove that the following identity holds by structural induction:

$$\#_a(t \cdot s) = \#_a(t) + \#_a(s).$$

To do this, we apply structural induction on t. The base case is when t is λ, in which case we have $\#_a(\lambda \cdot s) = \#_a(s) = 0 + \#_a(s) = \#_a(\lambda) + \#_a(s)$. Now for the inductive step, let $t = \langle x, u\rangle$ where x is a symbol. We need to consider the two cases that x is, or is not a.

$x = a$: By the inductive definitions of $\cdot$ and $\#_a$ we have

$$\#_a(\langle a, u\rangle \cdot s) = \#_a(\langle a, u \cdot s\rangle) = \#_a(u \cdot s) + 1 \quad \text{and} \quad \#_a(\langle a, u\rangle) + \#_a(s) = \#_a(u) + 1 + \#_a(s).$$

So by using the inductive hypothesis $\#_a(u \cdot s) = \#_a(u) + \#_a(s)$, we are done, i.e.

$$\#_a(\langle a, u\rangle \cdot s) = \#_a(u \cdot s) + 1 = \#_a(u) + \#_a(s) + 1 = \#_a(\langle a, u\rangle) + \#_a(s)$$

$x \neq a$: Again by the inductive definitions we have

$$\#_a(\langle x, u\rangle \cdot s) = \#_a(\langle x, u \cdot s\rangle) = \#_a(u \cdot s) \quad \text{and} \quad \#_a(s) + \#_a(\langle x, u\rangle) = \#_a(u) + \#_a(s).$$

So by using the inductive hypothesis we are done.

Thus we have that the identity $\#_a(t \cdot s) = \#_a(t) + \#_a(s)$ holds for all strings t and s. □

Exercise 4. *Lewis–Zax: Exercise 8.4.*
Recall that $|s|$ here means the length *of the string, as defined (inductively) in Lewis–Zax chapter 8.*

Solution. We will use structural induction on u to prove that $|t \cdot s| = |t| + |s|$. For the base case just observe $|\lambda \cdot s| = |s| = 0 + |s| = |\lambda| + |s|$. For the constructor case, let $s = \langle x, u \rangle$ where $x \in \Sigma$, i.e. x is a symbol. We need to show that $|\langle x, u \rangle \cdot s| = |\langle x, u \rangle| + |s|$.
We have the following chain of equalities

$$|\langle x, u \rangle \cdot s| = |\langle x, u \cdot s \rangle| = |u \cdot s| + 1 = |u| + 1 + |s| = |\langle x, u \rangle| + |s|,$$

the first equality is by the inductive definition of $\cdot$, the second equality is by the inductive definition of $|-|$, the third equality is by the inductive hypothesis, and the fourth is again by the inductive definition of $|-|$. Hence $|u \cdot v| = |u| + |v|$ is true for all strings u, v. □

Exercise 5. *Lewis–Zax: Exercise 13.3.*
Refer to the chapter summary if any of the terminology used in this exercise is unfamiliar to you.

Solution.

(a)

	in-degree	out-degree
a	1	2
b	1	2
c	2	2
d	2	2
e	2	0

(b) In addition to the 5 trivial cycles (i.e. the cycles of length 0), there are 3 nontrivial cycles in G up to choice of starting vertex and they are as follows:

$$\{\{(a \to c \to d \to b \to a), (a \to d \to b \to a), (b \to c \to d \to b)\}.$$

The number of cycles that pass through each vertex is as follows:

a	b	c	d	e
3	4	3	4	1

(c) The distances $d_G(\text{row}, \text{column})$ are detailed in the following table:

	a	b	c	d	e
a	0	2	1	1	2
b	1	0	1	2	2
c	3	2	0	1	1
d	2	1	2	0	1
e	∞	∞	∞	∞	0

(d) Since the graph has 5 vertices, the longest path can be at most 4 arcs long. One path that is 4 arcs long is $a \to d \to b \to c \to e$, so the longest path is exactly 4 arcs long. All of the paths of length 4 are

$$\{(a \to d \to b \to c \to e), (b \to a \to c \to d \to e), (d \to b \to a \to c \to e)\}.$$ □