MA0301
ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2023

Solutions to Exercise Set 1

Exercise 1. *Write out the truth table for the following compound statements:*

a) $r \Rightarrow (p \Rightarrow q)$ *b)* $p \Rightarrow (q \vee r)$ *c)* $p \Rightarrow ((q \Rightarrow \neg r) \vee (p \oplus r))$

Can you make the statements in a) and b) logically equivalent by adding a single negation?

Solution. a) The truth table is the following:

p	q	r	$p \Rightarrow q$	$r \Rightarrow (p \Rightarrow q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	T	T
F	F	T	T	T
T	T	F	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

b) The truth table is the following:

p	q	r	$q \vee r$	$p \Rightarrow (q \vee r)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T
T	T	F	T	T
T	F	F	F	F
F	T	F	T	T
F	F	F	F	T

c) The truth table is the following:

p	q	r	$\neg r$	$q \Rightarrow \neg r$	$p \oplus r$	$(q \Rightarrow \neg r) \vee (p \oplus r)$	$p \Rightarrow ((q \Rightarrow \neg r) \vee (p \oplus r))$
T	T	T	F	F	F	F	F
T	F	T	F	T	F	T	T
F	T	T	F	F	T	T	T
F	F	T	F	T	T	T	T
T	T	F	T	T	T	T	T
T	F	F	T	T	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	F	T	T

Changing the expression in a) to $\neg r \Rightarrow (p \Rightarrow q)$ makes it logically equivalent to the compound statement from b). □

Exercise 2. *Let p, q, r denote the following statements about a particular triangle ABC.*

p : *Triangle ABC is isosceles.*
q : *Triangle ABC is equilateral.*
r : *Triangle ABC is equiangular.*

Translate each of the following into an English sentence.

a) $q \Rightarrow p$ *b)* $\neg p \Rightarrow \neg q$ *c)* $q \Leftrightarrow r$ *d)* $p \wedge \neg q$ *e)* $r \Rightarrow p$

Solution.

a) If the triangle ABC is equilateral then ABC is isosceles.
b) If the triangle ABC is not isosceles then ABC is not equilateral.
c) The triangle ABC is equilateral if, and only if, ABC is equiangular.
d) The triangle ABC is isosceles and not equilateral.
e) If the triangle ABC is equiangular then ABC is isosceles. □

Exercise 3. *Let p, q and r be propositional variables. Verify whether the following compound statements are tautologies, satisfiable, or unsatisfiable.*

a) $(p \vee q) \vee (p \Rightarrow q)$ *b)* $(p \Rightarrow (q \wedge \neg q)) \wedge p$ *c)* $(p \Rightarrow (q \wedge \neg q)) \wedge p \Rightarrow r$

Hint: *You may use truth tables but it is also possible to write out shorter arguments in sentences.*

Solution.

a) The implication $(p \Rightarrow q)$ can only have the truth value F when p has the value T. But in this case $(p \vee q)$ always has the truth value T. It follows that the whole compound statement always has the truth value T (that is for any choice of truth values for p and q) and is thus a tautology.
b) We argue that the whole compound statement can never be true and is thus unsatisfiable. To this end, first observe that the statement $(q \wedge \neg q)$ is always false (unsatisfiable). For this reason $(p \Rightarrow (q \wedge \neg q))$ will only be true if p is false. Thus the whole compound statement can never be true.
c) From b) we know that $(p \Rightarrow (q \wedge \neg q)) \wedge p$ is unsatisfiable and thus always false. It follows that the implication $(p \Rightarrow (q \wedge \neg q)) \wedge p \Rightarrow r$ is always true (regardless of the truth value r has) and is thus a tautology. □

Exercise 4. *Let the variable r have truth value T. Determine all truth value assignments for the propositional variables p, q and s for which the truth value of the following statement is T:*

$$(p \Rightarrow (q \wedge r)) \wedge (r \Rightarrow ((s \vee \neg q) \wedge p))$$

Solution. First of all, we observe that for the whole compound statement to be true we need that

(1) $(p \Rightarrow (q \wedge r))$ is true, and
(2) $(r \Rightarrow ((s \vee \neg q) \wedge p))$ is true.

Because r is true, the second of these statements can only be true if $((s \vee \neg q) \wedge p)$ is true. Thus p needs to have the truth value T. We can use this fact about p, together with requirement 1 to conclude that $(q \wedge r)$ must be true. Thus q also needs to have the truth value T. Because we know that $(s \vee \neg q)$ needs to be true, it follows now that also s has to have the truth value T. This shows that if the whole compound statement and r has the truth value T, then p, q, and s have the truth value T as well. Thus the only choice of truth value assignments for p, q and s is that they are all T. □

Exercise 5. *Prove the distributive laws of propositional logic using truth tables:*

$$a)\ \ p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \qquad b)\ \ p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r).$$

Solution.

a) For the first law, we have the truth table

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	F	T	T	T	F	T	T
F	T	T	T	F	F	F	F
F	F	T	T	F	F	F	F
T	T	F	T	T	T	F	T
T	F	F	F	F	F	F	F
F	T	F	T	F	F	F	F
F	F	F	F	F	F	F	F

We see that for every possible configuration of p, q, and r that the two sides of the proposed law agree, hence they are logically equivalent.

b) We follow the same procedure as in a). The truth table is

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T
F	F	T	F	F	F	T	F
T	T	F	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	F	F	F	T	F	F
F	F	F	F	F	F	F	F

□

Exercise 6. *Use the laws of logic to simplify the statement* $(p \wedge q) \vee \neg p$.

Solution. Using the laws of logic we have:

$$\begin{aligned} (p \wedge q) \vee \neg p &\equiv \neg p \vee (p \wedge q) && \text{(Commutative law)} \\ &\equiv (\neg p \vee p) \wedge (\neg p \vee q) && \text{(Distributive law)} \\ &\equiv \neg p \vee q && \text{(Using absorption and that } (\neg p \vee p) \text{ is a tautology)} \end{aligned}$$

(This can also be rewritten as $p \Rightarrow q$.) □

Exercise 7. *Negate each of the following statements and simplify the result.*

$$a)\ (p \wedge q) \Rightarrow (\neg r \vee \neg s) \qquad b)\ p \Rightarrow (r \oplus s)$$

Solution.

a) Recall the equivalence $a \Rightarrow b \equiv \neg a \vee b$ and DeMorgan's laws $\neg(a \vee b) \equiv \neg a \wedge \neg b$ and $\neg(a \wedge b) \equiv \neg a \vee \neg b$. Thus we have the equivalences

$$\neg((p \wedge q) \Rightarrow (\neg r \vee \neg s)) \equiv \neg(\neg(p \wedge q) \vee (\neg r \vee \neg s)) \equiv (p \wedge q) \wedge \neg(\neg r \vee \neg s) \equiv p \wedge q \wedge r \wedge s$$

b) Similarly to a) but now also using the logical equivalence $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$ we get

$$\begin{aligned} &\neg(p \Rightarrow (r \oplus s)) \equiv \neg(\neg p \vee ((r \wedge \neg s) \vee (\neg r \wedge s))) \equiv p \wedge \neg(r \wedge \neg s) \wedge \neg(\neg r \wedge s) \\ &\equiv p \wedge (\neg r \vee s) \wedge (r \vee \neg s) \equiv p \wedge (r \Leftrightarrow s) \end{aligned}$$

□

Exercise 8. *Lewis, Zax: Exercise 9.6 (a).*

Solution. Using the propositions $p =$ "I study", $q =$ "I will pass the course", and $r =$ "The professor accepts bribes", translate the following into statements of propositional logic:

(1) If I do not study, then I will only pass the course if the professor accepts bribes: $\neg p \Rightarrow (q \Rightarrow r)$.
(2) If the professor accepts bribes, then I do not study: $r \Rightarrow \neg p$.
(3) The professor does not accept bribes, but I study and will pass the course: $\neg r \wedge (p \wedge q)$.
(4) If I study, the professor will accept bribes and I will pass the course: $p \Rightarrow (r \wedge q)$.
(5) I will not pass the course but the professor accepts bribes: $\neg q \wedge r$. □

Exercise 9. *Let p be an arbitrary proposition, and let q and r be the following statements:*

q : *The proposition p is a tautology.*
r : *The proposition p is satisfiable.*

Determine whether the following statements are tautologies, satisfiable or unsatisfiable:

$$a)\ q \Rightarrow r \qquad b)\ r \qquad c)\ \neg q \Rightarrow \neg r$$

Solution.

a) Every tautology is, in particular, satisfiable, so the only problematic case of the implication cannot occur. Hence $q \Rightarrow r$ is a tautology.

b) If p is any satisfiable proposition, then r is true, therefore r is satisfiable. However, not all propositions are satisfiable, so r is not a tautology.

c) If p is an unsatisfiable proposition, then, in particular, it is not a tautology, hence the statement $\neg q \Rightarrow \neg r$ is satisfiable. However, if p is some satisfiable proposition that is not a tautology, then the statement $\neg q \Rightarrow \neg r$ is not satisfied, hence it is not a tautology. □