MA0301
ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2022

Set 6 Solutions

Exercise 1. *Lewis, Zax: Exercise 6.3a.*

Solution.
Given that r is an injective function from A to B we know that $f(x) = f(y)$ *where* $x, y \in A$ *implies* $x = y$ *(the definition of injectivity), i.e. that each image of an element from* $a \in A$ *is a unique element* $f(a) \in B$*. Let's say there are* $|A| = n$ *elements in A, then we know there are n unique images in B corresponding to the n elements in A. Because* $|B| = |A| = n$ *this implies that for each of the n elements* $b \in B$ *there is some* $a \in A$ *such that* $f(a) = b$*, which is exactly the definition of surjectivity. Then, if r is both injective and surjective, it is also bijective.*

Exercise 2. *Lewis, Zax: Exercise 6.7.*

Solution.
$f^{-1} = \frac{n}{2}$ *and* $g^{-1} = \frac{n-1}{2}$.
We will call $\mathbb{Z}$ *set A, the even integers set B, and the odd integers set C. Then we have that* $f : A \to B$ *and* $g : A \to C$*. Both f and g are bijections. We want to find a function h such that* $h : B \to C$ *and h is bijective. By theorem 6.4 this function exists and can be found using the formula* $h(n) = g(f^{-1}(n))$.
$h(n) = g(\frac{n}{2}) = 2(\frac{n}{2}) + 1 = n + 1$*. Then* $h(n) = n + 1$ *which clearly maps the even integers to the odd integers and is bijective as desired.*

Exercise 3. *Use induction to show that: if n is a positive integer, then* $\sum_{m=1}^{n} m = 1+2+3+...+n = \frac{n(n+1)}{2}$.

Base Case:
Let $n = 1$*. Then* $\sum_{m=1}^{1} m = 1 = \frac{1(1+1)}{2}$.
Inductive Hypothesis:
Assume this is true up to $n = k$, $\sum_{m=1}^{k} m = 1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}$.
Induction:
We want to show this is true for $n = k + 1$*, that* $\sum_{m=1}^{k+1} m = 1 + 2 + 3 + ... + k + (k + 1) = \frac{(k+1)((k+1)+1)}{2} = \frac{(k+1)(k+2)}{2}$*, based on our assumption.*
Starting with our assumption $\sum_{m=1}^{k} m = 1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}$*, we can add a term of (k+1) to both sides of the equation to get* $(\sum_{m=1}^{k} m) + (k + 1) = \frac{k(k+1)}{2} + (k + 1)$*, which is equivalent to* $\sum_{m=1}^{k+1} m = \frac{k(k+1)}{2} + (k + 1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$.

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$\frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$.
Then $\sum_{m=1}^{k+1} m = \frac{(k+1)(k+2)}{2}$ *and* $\sum_{m=1}^{n} m = 1+2+3+...+n = \frac{n(n+1)}{2}$ *when* n *is a positive integer.* □

Exercise 4. *Lewis, Zax: Exercise 3.7.*
Base Case:
Let $n = 1$. *Then* $\sum_{i=1}^{1} i^3 = 1^3 = (1)^2 = (\sum_{i=1}^{1} i)^2$.
Inductive Hypothesis:
Assume this is true up to $n = k$, $\sum_{i=1}^{k} i^3 = (\sum_{i=1}^{k} i)^2$.

Induction step: we will prove that the statement is also valid for $k+1$. Splitting the sum and using the induction hypothesis we have that

$$\begin{aligned}
\left(\sum_{i=1}^{k+1} i\right)^2 &= \left(\left(\sum_{i=1}^{k} i\right) + (k+1)\right)^2 \\
&= \left(\sum_{i=1}^{k} i\right)^2 + 2\left(\sum_{i=1}^{k} i\right)(k+1) + (k+1)^2 \\
&= \sum_{i=1}^{k} i^3 + 2\left(\sum_{i=1}^{k} i\right)(k+1) + (k+1)^2 \qquad \text{(by induction hypothesis)} \\
&= \sum_{i=1}^{k} i^3 + (k+1)\left(2\left(\sum_{i=1}^{k} i\right) + k + 1\right) \\
&= \sum_{i=1}^{k} i^3 + (k+1)\left(2\frac{k(k+1)}{2} + k + 1\right) \qquad \text{by Gauss Formula } \textstyle\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \\
&= \sum_{i=1}^{k} i^3 + (k+1)(k(k+1) + (k+1)) \\
&= \sum_{i=1}^{k} i^3 + (k+1)(k+1)^2 \\
&= \sum_{i=1}^{k} i^3 + (k+1)^3 \\
&= \sum_{i=1}^{k+1} i^3,
\end{aligned}$$

that is what we wanted to show. Hence, by mathematical induction, we conclude that for any $n \geq 0$, $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$.

Exercise 5. *a) Find a formula for* $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n(n+1)}$ *by examining the values of this expression for small values of* n.

Solution.
Examining the values for $n = 1, 2, 3...$ *we can derive the formula that for all* $n > 0$, $\sum_{i=1}^{n} \frac{1}{i\cdot(i+1)} = \frac{n}{n+1}$.

b) Use induction to prove the formula you conjectured in part a.
Solution.

Base Case:
Let $n = 1$. *Then* $\sum_{i=1}^{1} \frac{1}{i\cdot(i+1)} = \frac{1}{1\cdot 2} = \frac{1}{1+1} = \frac{n}{n+1}$.
Inductive Hypothesis:
Assume this is true up to $n = k$, $\sum_{i=1}^{k} \frac{1}{i\cdot(i+1)} = \frac{k}{k+1}$.
Induction:
We want to show this is true for $n = k+1$, *that* $\sum_{i=1}^{k+1} \frac{1}{i\cdot(i+1)} = \frac{k+1}{k+2}$.
Starting with our assumption, $\sum_{i=1}^{k} \frac{1}{i\cdot(i+1)} = \frac{k}{k+1}$, *we will add a term of* $\frac{1}{(k+1)(k+2)}$ *to each side of the equation:*
$(\sum_{i=1}^{k} \frac{1}{i\cdot(i+1)}) + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$. *This is equivalent to:*
$\sum_{i=1}^{k+1} \frac{1}{i\cdot(i+1)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$.
Then we will simplify $\frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$:
$\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$.
Then $\sum_{i=1}^{k+1} \frac{1}{i\cdot(i+1)} = \frac{k+1}{k+2}$ *and for all* $n > 0$, $\sum_{i=1}^{n} \frac{1}{i\cdot(i+1)} = \frac{n}{n+1}$. □

Exercise 6. *What is wrong with this "proof"?*
"Theorem" For every positive integer n, if x and y are positive integers with $max(x, y) = n$, *then* $x = y$.
Basis Step: Suppose that $n = 1$. *If* $max(x, y) = 1$ *and x and y are both positive integers we have* $x = y$.
Inductive Step: Let k be a positive integer. Assume that whenever $max(x, y) = k$ *and x and y are positive integers, then* $x = y$. *Now let* $max(x, y) = k + 1$, *where x and y are positive integers. Then* $max(x - 1, y - 1) = k$, *so by the inductive hypothesis,* $x - 1 = y - 1$. *It follows that* $x = y$, *completing the inductive step.*

Solution.

In this case we know that x and y are positive integers but we don't know which positive integers. Meaning we do not know if $x - 1$ *or* $y - 1$ *is a positive integer. If we do not know that* $x - 1$ *or* $y - 1$ *are both positive integers then our assumption no longer applies, and the proof is not valid.*

Exercise 7. *Use induction to prove that* $3^n < n!$ *if n is an integer greater than 6.*

Solution.
Base Case:
Let $n = 7$. *Then* $3^7 < 7! = 2187 < 5040$.
Inductive Hypothesis:
Assume this is true up to $n = k$, $3^k < k!$.

Induction:
We want to show this is true for $n = k+1$, *that* $3^{k+1} < (k+1)!$.
We have that $3^k < k!$, *then we also have that* $3 \cdot 3^k < 3 \cdot k!$. *We know that* $k+1 > 3$, *so* $3 \cdot 3^k < (k+1) \cdot k!$ *which is equivalent to* $3^{k+1} < (k+1)!$.
Then by induction $3^n < n!$ *is true for all* $n > 6$. □

Exercise 8. *Use induction to prove that* 6 *divides* $n^3 - n$ *whenever* n *is a nonnegative integer.*

Solution.
Base Case:
Let $n = 0$. *Then* $0^3 - 0 = 0$ *and* $6|0$.
Inductive Hypothesis:
Assume this is true up to $n = k$, $6|k^3 - k$.
Induction:
We want to show this is true for $n = k+1$, *that* $6|(k+1)^3 - (k+1)$.
We will expand and rearrange $(k+1)^3 - (k+1)$.
$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 =$
$k^3 - k + 3k^2 + 3k = (k^3 - k) + 3k(k+1)$. *From our inductive hypothesis we have that* $k^3 - k$ *is divisible by 6. Examining the second term we see there is a factor of 3 present. Then in order for our second term to be divisible by 6 we also need a factor of 2. We have both of factor of* k *and* $k+1$. *Regardless of the value of* k *one of these factors is even, meaning there is also a guaranteed factor of 2 in the second term. Then both terms are divisible by 6 and their sum is also,* $6|(k+1)^3 - (k+1)$. *Then for all* $n \geq 0$, 6 *divides* $n^3 - n$. □