MA0301
ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2022

Solutions Set 4

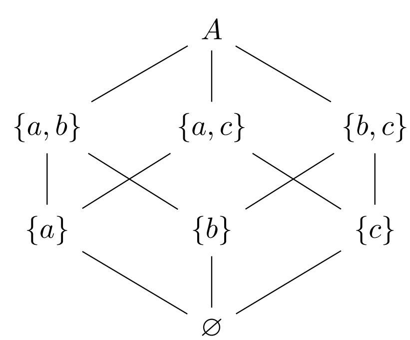
Exercise 1. *Consider the set* $X = \{a, b, c\}$.

a) What is the power set $\mathcal{P}(X)$*?*
b) Show that for any set Y*, the relation defined by set inclusion* $R = \{\langle A, B\rangle | A \subseteq B\}$ *defines a partial ordering on* $\mathcal{P}(Y)$.
c) Draw the Hasse diagram for $\mathcal{P}(X)$ *with the partial ordering given by set inclusion like in b).*

Solution.

a) It's $\mathcal{P}(X) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$
b) We have to check the three defining properties of a partial ordering.
  - *Reflexivity*: For all $A \subset X$, it is true that $A = A$ and thus $A \subseteq A$.
  - *Transitivity*: Let $A \subseteq X$, $B \subseteq X$ and $C \subseteq X$ such that $A \subseteq B$ and $B \subseteq C$. For $a \in A$ we know that because of $A \subseteq B$ also $a \in B$. Then from $B \subseteq C$ it follows that also $a \in C$. Consequently we have $A \subseteq C$.
  - *Antisymmetry*: If for $A \subseteq X$ and $B \subseteq X$ we have that both i) $A \subseteq B$ and ii) $B \subseteq A$ then, if $a \in A$ it follows from i) that also $a \in B$. If on the other hand $b \in B$ it follows with ii) that also $b \in A$. Consequently we have $A = B$.

c)



Exercise 2. *Consider the set* $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. *Define the relation* R *on* X *that relates every number in* X *to those that have the same number of divisors as it.*

a) Show that R *is an equivalence relation.*
b) Find the partition of X *corresponding to* R.

Solution.

a) We shall prove that R is reflexive, symmetric and transitive.

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- *Reflexivity:* Take $x \in X$. Obviously, x has the same number of divisors as x, thus for all $x \in X$, we have $(x, x) \in R$. *Symmetry:* Let $x, y \in X$ such that $(x, y) \in R$. By definition of R, we have that y has the same number of divisors as x. This is equivalent to say that x has the same number of divisors as y. Again by definition of R, we have that $(y, x) \in R$. *Transitivity:* Let $x, y, z \in X$ such that $(x, y), (y, z) \in R$. By definition of R, we have that y has the same number of divisors as x, and z has the same number of divisors as y. The two latter statements imply that z has the same number of divisors as x. Hence $(x, z) \in R$.

From the above, we conclude that R is an equivalence relation.

b) Recall that two elements in X belong to the same block of the partition induced by R (i.e. the same equivalence class) if and only if they have the same number of divisors. By counting the number of divisors of the elements in S, we have
- 1 has only one divisor.
- 2, 3, 5, 7 and 11 are prime numbers and thus they have 2 divisors.
- 4 and 9 are of the form p^2 for some prime number p, and thus they have 3 divisors.
- 6 and 10 are of the form pq for some prime numbers p and q. They have 4 divisors. Also, 8 is a cube of the form p^3 for $p = 2$, and so it has 4 divisors.
- Finally, 12 is of the form $2^2 \cdot 3$, and it has 6 divisors $(1, 2, 3, 4, 6, 12)$.

Hence, the partition is the following: $\{\{1\}, \{2, 3, 5, 7, 11\}, \{4, 9\}, \{6, 8, 10\}, \{12\}\}$.

Exercise 3. *Let $p \in \mathbb{Z}$. Define the following relation*

$$R_p = \{\langle x, y\rangle | \exists n \in \mathbb{Z} : x = y + n \cdot y\} \subset \mathbb{Z} \times \mathbb{Z}.$$

a) Show that R is an equivalence relation.

b) Characterize the equivalence classes of R_p. How many are there for a fixed p?.

Solution.

a) Again we show that R_p is reflexive, symmetric and transitive.
- *Reflexivity* Clearly $x = x + n \cdot p$ for $n = 0$.
- *Symmetry* If $(x, y) \in R_p$, then there exists a $n \in \mathbb{Z}$ such that $x = y + n \cdot p$. We then also have $y = x + n' \cdot p$ with $n' = -n$.
- *Transitivity* If $(x, y) \in R_p$ and $(y, z) \in R_p$, then there exist $n_1, n_2 \in \mathbb{Z}$ such that $x = y + n_1 \cdot p$ and $y = z + n_2 \cdot p$. With setting $n = n_1 + n_2$ we then have $x = z + n \cdot p$.

b) There are p equivalence classes characterized by the representatives $0, 1 \ldots p-1$. The class $[i]$ consists of $[i] = \{\ldots, i - 2 \cdot p, i - p, i, i + p, i + 2 \cdot p, \ldots\}$.

Exercise 4. *a) Let X be a set and let $R_1 \subseteq X \times X$ and $R_2 \subseteq X \times X$ be two equivalence relations on X. Show that $R_1 \cap R_2$ also defines an equivalence relation.*

b) Let the cardinality of X be finite. Define $R \subseteq X \times X$ to be any relation on X. We make a list with all equivalence relations that contain R:

$$\{R_1, R_2, \ldots, R_n\} = \{R_i \subseteq X \times X | (R \subseteq R_i) \wedge (R_i \textit{ is an equivalence relation})\}.$$

Note that this list is finite because X has finite cardinality. Prove that $\tilde{R} := R_1 \cap R_2 \cap \ldots \cap R_n$ is the smallest equivalence relation on X that contains R i.e that for all equivalence relations $A \subset X \times X$ such that $R \subseteq A$ it follows that $\tilde{R} \subseteq A$. We call $\tilde{R}$ the equivalence relation generated by R.

c) *What is the equivalence relation generated by the relation from exercise 1 a)?*

Solution.

a) Again we show the three defining properties of an equivalence relation.
   - *Reflexivity* For all $x \in X$ we know that $(x, x) \in R_1$ and $(x, x) \in R_2$ because both are equivalence relations and thus satisfy reflexivity. Thus (x, x) is also an element of $R_1 \cap R_2$.
   - *Symmetry* If $(x, y) \in R_1 \cap R_2$ we know that $(x, y) \in R_1$ and $(x, y) \in R_2$. But because both R_1 and R_2 are equivalence relations they have to satisfy symmetry and thus also $(y, x) \in R_1$ and $(y, x) \in R_2$. Consequently we have $(y, x) \in R_1 \cap R_2$.
   - *Transitivity* Let $(x, y) \in R_1 \cap R_2$ and $(y, z) \in R_1 \cap R_2$. Then both of these elements are also in R_1 and R_2. By transitivity of the latter two relations we get $(x, z) \in R_1$ and $(x, z) \in R_2$. Thus also $(x, z) \in R_1 \cap R_2$.

b) Let $A \subseteq X \times X$ be any equivalence relation such that $R \subseteq A$. Then it follows that A is part of the list of all equivalence relations that contain R, i.e there is some $i \in \{1, \ldots, n\}$ such that $A = R_i$. Consequently $\tilde{R} \subset A$.

c) By adding all the necessary elements to R so that the relation becomes symmetric and at the same time stays transitive we quickly end up with all elements in $\mathcal{P}(X)$ being equivalent to one another. This also follows by observing that the Hasse diagram of the relation is path connected (every two vertices in it are connected by a path).