MA0301
ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2022

Solutions Set 3

Exercise 1. *Write the following logical statement in prenex normal form:*

$$\neg\forall x(P(x) \Rightarrow \exists y Q(x,y))$$

Solution. We use the rules $\forall x\neg F \equiv \neg\exists x F$, $\exists x \neg F \equiv \neg\forall x F$ and the rule of scope change.

$$\begin{aligned} &\neg\forall x(P(x) \Rightarrow \exists y Q(x,y)) \\ =&\exists x\neg(P(x) \Rightarrow \exists y Q(x,y)) \\ =&\exists x\neg\exists y(P(x) \Rightarrow Q(x,y)) \\ =&\exists x\forall y\neg(P(x) \Rightarrow Q(x,y)) \end{aligned}$$

Exercise 2. *What is the power set of* $A := \{\{a\},\{b,c\},\{c,d,\{e,f\}\}\}$

Solution.

$$\begin{aligned} \mathcal{P}(A) = \{\varnothing,&\{\{a\}\},\{\{b,c\}\},\{\{c,d,\{e,f\}\}\},\{\{a\},\{b,c\}\} \\ &,\{\{a\},\{c,d,\{e,f\}\}\},\{\{b,c\},\{c,d,\{e,f\}\}\},A\} \end{aligned}$$

Exercise 3. *In the universe of real numbers, let* $A = \{x|1 \leq x \leq 5\}$ *and* $B = \{x|3 \leq x \leq 7\}$*.*

a) What is the intersection $A \cap B$*?*
b) Let $A \triangle B = \{x | x \in A \cup B \wedge x \notin A \cap B\}$*. This is also called the symmetric difference of* A *and* B*. Find sets* C *and* D *such that* $A \triangle B = C \cup D$*.*

Solution.

a) The intersection is $A \cap B = \{x|3 \leq x \leq 5\}$
b) Pick $C = \{x|1 \leq x \leq 3\}$ and $D = \{x|5 \leq x \leq 7\}$.

Exercise 4. *Let* X *and* Y *be two sets. Prove that* $\overline{X - Y} = \overline{X} \cup Y$*.*

Solution. In general the statement $\overline{A} = \overline{B}$ is equivalent to the statement $A = B$. We observe that

$$\overline{X} \cup Y = \overline{\overline{\overline{X} \cup Y}} = \overline{X \cap \overline{Y}}.$$

The statement $\overline{X - Y} = \overline{X} \cup Y$ is thus equivalent to $X - Y = X \cap \overline{Y}$, which in turn is true by definition.

Exercise 5. *For two sets* X *and* Y *prove that*

a) $(\overline{X} \cup Y) \cap (X \cup Y) = Y$
b) $(\overline{X} \cap Y) \cup (X \cap Y) = Y$

Solution. We use the distributive laws for union and intersection.

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a) $(\overline{X} \cup Y) \cap (X \cup Y) = (\overline{X} \cap X) \cup Y = \varnothing \cup Y$
b) $(\overline{X} \cap Y) \cup (X \cap Y) = (\overline{X} \cup X) \cap Y = \mathcal{U} \cap Y = Y$

Exercise 6. *Prove that the following three statements are equivalent:*

$$i)\ \overline{Y} - X = \overline{Y}, \quad ii)\ X \subseteq Y \quad iii)\ X \cap Y = X$$

Solution. We have to show $i) \Leftrightarrow ii) \Leftrightarrow iii)$. By Law of Syllogism, this is equivalent to showing $i) \Rightarrow ii) \Rightarrow iii) \Rightarrow i)$.

- $(i) \Rightarrow ii))$ We assume that $i)$ is true and show that then $ii)$ must be true as well. If $a \in X$, then it follows from $i)$ that it is not in $\overline{Y}$. Consequently $a \in Y$ and it follows that $ii)$ is true.
- $(ii) \Rightarrow iii))$ We assume that $ii)$ is true and show that then $iii)$ must also be true. If $a \in X \cap Y$, then we also have $a \in X$. If on the other hand $a \in X$ then we can use that because of $X \subseteq Y$ also $a \in Y$ and thus $a \in X \cap Y$.
- $(iii) \Rightarrow i))$ Now we assume that $iii)$ is true. If $a \in \overline{Y} - X$, then also $a \in \overline{Y}$. On the other hand if $a \in \overline{Y}$, then a is not in X because $\overline{X} \cup \overline{Y} = \overline{X \cap Y} = \overline{X}$. Thus $a \in \overline{Y} - X$.

Exercise 7. *Lewis, Zax: Exercise 5.4*

Solution The cardinality of the power set of A is $\mathcal{P}(\mathcal{A}) = 2^n$. The reason is that every element of A can either be or not be in a given subset of A. Thus every element in A contributes a factor of two to the number of all possible subsets. We thus have that $|\mathcal{P}(A) - \{\{x\}|x \in A\}| = 2^n - n$.

Exercise 8. *Lewis, Zax: Exercise 5.5*

Solution

a) We have that $|A \times B| = |A| \cdot |B|$. Using the formula $\mathcal{P}(\mathcal{X}) = 2^{|\mathcal{X}|}$, which is true for every finite set X we find that $|\mathcal{P}(A \times B)| = 2^{|A| \cdot |B|}$. On the other hand is $|\mathcal{P}(A)| \cdot |\mathcal{P}(B)| = 2^{|A|} \cdot 2^{|B|}$. We find
$$\frac{|\mathcal{P}(A \times B)|}{|\mathcal{P}(A)| \cdot |\mathcal{P}(B)|} = \frac{2^{|A| \cdot |B|}}{2^{|A|} \cdot 2^{|B|}} = \frac{2^{|A| \cdot |B|}}{2^{|A| + |B|}}$$
Thus
   - $|\mathcal{P}(A)| \cdot |\mathcal{P}(B)|$ is larger than $|\mathcal{P}(A \times B)|$ if the cardinality of A or B is one or either of the two is the empty set.
   - They are the same if $|A| = 2$ and $|B| = 2$, or if both A and B are the empty set.
   - $|\mathcal{P}(A \times B)|$ is larger than $|\mathcal{P}(A)| \cdot |\mathcal{P}(B)|$ in all the other cases.

b) We use the distributive law for intersections and show:
$$\begin{aligned} &(A - B) \cap (B - A) \\ =&(A \cap \overline{B}) \cap (B \cap \overline{A}) \\ =&A \cap (\overline{B} \cap (B \cap \overline{A})) \\ =&A \cap ((\overline{B} \cap B) \cap \overline{A}) \\ =&A \cap (\varnothing \cap \overline{A}) \\ =&A \cap \varnothing \\ =&\varnothing \end{aligned}$$
The statement is thus true.