MA0301
ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2022

Solutions Set 1

Exercise 1. *Write out the truth table for the following compound statements:*

$$a)p \Rightarrow (q \vee r) \quad b)r \Rightarrow (p \Rightarrow q) \quad c)p \Rightarrow (q \Rightarrow \neg r) \vee (p \oplus r)$$

Can you make the statements in a) and b) logically equivalent by just adding one single negation?

Solution. a) The truth table is the following:

p	q	r	$q \vee r$	$p \Rightarrow (q \vee r)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T
T	T	F	T	T
T	F	F	F	F
F	T	F	T	T
F	F	F	F	T

b) The truth table is the following:

p	q	r	$p \Rightarrow q$	$r \Rightarrow (p \Rightarrow q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	T	T
F	F	T	T	T
T	T	F	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

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c) The truth table is the following:

p	q	r	$\neg r$	$q \Rightarrow \neg r$	$p \oplus r$	$(q \Rightarrow \neg r) \vee (p \oplus r)$	$p \Rightarrow (q \Rightarrow \neg r) \vee (p \oplus r)$
T	T	T	F	F	F	F	F
T	F	T	F	T	F	T	T
F	T	T	F	F	T	T	T
F	F	T	F	T	T	T	T
T	T	F	T	T	T	T	T
T	F	F	T	T	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	F	T	T

By changing the the expression in b) to $\neg r \Rightarrow (p \Rightarrow q)$ it becomes logically equivalent to the compound statement from a).

Exercise 2. *Let p, q and r be propositional variables. Verify whether the following compound statements are satisfiable, tautologies or unsatisfiable. Hint: You may use truth tables but it is also possible to write out shorter arguments in sentences.*

$$a)(p \vee q) \vee (p \Rightarrow q) \quad b)[p \Rightarrow (q \wedge \neg q)] \wedge p \quad c)[p \Rightarrow (q \wedge \neg q)] \wedge p \Rightarrow r$$

Solution. a) The impication $(p \Rightarrow q)$ can only have the truth value F when p has the value T. But in this case $(p \vee q)$ always has the truth value T. It follows that the whole compound statement always has the truth value T (that is for any choice of truth values for p and q) and it is thus a tautology.
b) We argue that the whole compound statement can never be true and is thus unsatisfiable. To this end first observe that the statement $(q \wedge \neg q)$ is always false (unsatisfiable). For this reason $[p \Rightarrow (q \wedge \neg q)]$ will only be true if p is false. Thus the whole compound statement can never be true.
c) From b) we know that $[p \Rightarrow (q \wedge \neg q)] \wedge p \Rightarrow r$ is unsatisfiable and thus always false. It follows that the implication $[p \Rightarrow (q \wedge \neg q)] \wedge p \Rightarrow r \Rightarrow r$ is always true (regardless of the truth value r has) and is thus a tautology.

Exercise 3. *Let $a, b, c \in \mathbb{R}$ denote real numbers and consider the following statements about them*

(1) p: a is smaller than b.
(2) q: b is smaller than c.
(3) r: a is smaller than c.
(4) s: a is equal to b.
(5) t: b is equal to c.
(6) u: a is equal to c.

Translate the following into an English sentence and comment on whether they should be reasonable statements about real numbers.

$$a)p \wedge q \Rightarrow r \quad b)p \wedge q \Rightarrow u \quad c)(p \vee s) \wedge (q \vee t) \wedge u \Rightarrow s$$

Solution.

a) If a is smaller than b and b is smaller than c, then a is smaller than c. This statement is reasonable.
b) If a is smaller than b and b is smaller than c then a is equal to c. This statement is not reasonable.

c) If a is smaller or equal to b and b is smaller or equal to c and a is equal to c, then a is equal to b. This is a reasonable statement.

Exercise 4. *If the statement q has the truth value T, determine all truth value assignments for the propositional variables p, r and s for which the truth value of the statement*

$$(q \Rightarrow [(p \vee \neg r) \wedge s]) \wedge [s \Rightarrow (r \wedge q)]$$

is T.

Solution. First of all we observe that for the whole compound statement to be true we need that both

(1) $(q \Rightarrow [(p \vee \neg r) \wedge s])$ is true
(2) $[s \Rightarrow (r \wedge q)]$ is true.

Because q is true that first of these statements can only be true if $[(p \vee \neg r) \wedge s])$ is true. Thus s needs to have the truth value T. We can use this fact about s, together with requirement 1) to conclude that $(r \wedge q)$ must be true. Thus r needs to have the truth value T. Because we know that $(p \vee \neg r)$ needs to be true it follows now that also p has to have the truth value T. This shows that if the whole compound statement and q have the truth value T, then p, r and s have the truth value T as well. Thus the only choice of truth value assignments for p, r and s is that they are all T.

Exercise 5. *Negate each of the following and simplify the resulting statement*

$$a)(p \wedge q) \Rightarrow (\neg r \vee \neg s) \quad b)p \Rightarrow (r \oplus s)$$

Solution.

a) Recall the equivalence $a \Rightarrow b \equiv \neg a \vee b$ and the DeMorgan's laws $\neg(a \vee b) \equiv \neg a \wedge \neg b$ and $\neg(a \wedge b) \equiv \neg a \vee \neg b$. Hence

$$\neg[(p \wedge q) \Rightarrow (\neg r \vee \neg s)] \equiv \neg[\neg(p \wedge q) \vee (\neg r \vee \neg s)] \equiv (p \wedge q) \wedge \neg(\neg r \vee \neg s) \equiv p \wedge q \wedge r \wedge s$$

b) Similarly to a) but now also using the logical equivalence $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$ we get

$$\neg[p \Rightarrow (r \oplus s)] \equiv \neg[\neg p \vee [(r \wedge \neg s) \vee (\neg r \wedge s)]] \equiv p \wedge \neg(r \wedge \neg s) \wedge \neg(\neg r \wedge s)$$
$$\equiv p \wedge (\neg r \vee s) \wedge (r \vee \neg s) \equiv p \wedge (r \Leftrightarrow s)$$

Exercise 6. *Lewis, Zax: Exercise 9.3.*

Solution. Let p, q and r be propositional variables. We construct the following truth table

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

Since the fifth and the seventh columns are identical, then we have that $(p \vee q) \vee r \equiv p \vee (q \vee r)$.
Now, for the case of $\wedge$, we have the following truth table

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	F	T	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Since the fifth and the seventh columns are identical, then we have that $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$. $\square$

Exercise 7. *Lewis, Zax: Exercise 9.5. Using a truth table, determine whether each of the following compound propositions is satisfiable, a tautology, or unsatisfiable.*

$$a)\ p \Rightarrow (p \vee q), \quad b)\ \neg(p \Rightarrow (p \vee q)), \quad c)\ p \Rightarrow (p \Rightarrow q)$$

Solution. a) The truth table is the following:

p	q	$p \vee q$	$p \Rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Since the last column consists only of T's, we have that the proposition is a tautology.
b) Using the above table, we have

p	q	$p \Rightarrow (p \vee q)$	$\neg(p \Rightarrow (p \vee q))$
T	T	T	F
T	T	T	F
F	T	T	F
F	F	T	F

Since the last column consists only of T's, we have that the proposition is unsatisfiable.
c) The truth table is the following

p	q	$p \Rightarrow q$	$p \Rightarrow (p \Rightarrow q)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Since there is an assignment of truth values for p and q such that the compound statement is true, we have that the statement is satisfiable. $\square$

Exercise 8. *Lewis, Zax: Exercise 9.6 a.*

Solution. Using the propositions $p =$ "I study", $q =$ "I will pass the course", and $r =$ "The professor accepts bribes", translate the following into statements of propositional logic:

(1) If I do not study, then I will only pass the course if the professor accepts bribes: $\neg p \Rightarrow (q \Rightarrow r)$.
(2) If the professor accepts bribes, then I do not study: $r \Rightarrow \neg p$.
(3) The professor does not accept bribes, but I study and will pass the course: $\neg r \wedge (p \wedge q)$.
(4) If I study, the professor will accept bribes and I will pass the course: $p \Rightarrow (r \wedge q)$.
(5) I will not pass the course but the professor accepts bribes: $\neg q \wedge r$.