MA0301
ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2022

Set 7 Solutions

Exercise 1. *Lewis, Zax: Exercise 4.1.*

Solution.
Base cases.
$P(4) : a_4 = a_3 + a_1 = 9$ and $3|9$.
$P(5) : a_5 = a_4 + a_2 = 9 + 6 = 15$ and $3|15$.
$P(6) : a_6 = a_5 + a_3 = 15 + 9 = 24$ and $3|24$.
$P(7) : a_7 = a_6 + a_4 = 24 + 15 = 39$ and $3|39$.
Inductive hypothesis.
Let $n \geq 7$ be some fixed number and assume that for all m such that $1 \leq m \leq n$ it holds that a_m is divisible by 3 $(3|a_m)$.
Induction.
Show $P(n+1)$; that is, $a_{n+1} = a_n + a_{n-2}$ is divisible by 3.
$a_{n+1} = a_n + a_{n-2}$, we know that both a_n and a_{n-2} are divisible by 3 since both n, $n-2 \leq n$. Then we know the sum of a_n and a_{n-2} is divisible by 3 as well since we can write $a_n = 3k_1$ and $a_{n-2} = 3k_2$, for some $k_1, k_2 \in \mathbb{Z}$. Then $a_n + a_{n-2} = 3k_1 + 3k_2 = 3(k_1 + k_2) = a_{n+1}$. Then for all n, a_n is divisible by 3. □

Exercise 2. *Lewis, Zax: Exercise 4.2.*

Solution.
Base cases.
$P(8) : 8 \stackrel{?}{=} 3a + 5b$ for some $a, b \in \mathbb{Z}$. Let $a = -24$ and $b = 16$. Then $3a + 5b = -72 + 80 = 8$.
$P(9) : 9 \stackrel{?}{=} 3a + 5b$ for some $a, b \in \mathbb{Z}$. Let $a = -27$ and $b = 18$. Then $3a + 5b = -81 + 90 = 9$.
$P(10) : 10 \stackrel{?}{=} 3a + 5b$ for some $a, b \in \mathbb{Z}$. Let $a = -30$ and $b = 20$. Then $3a + 5b = -90 + 100 = 10$.
Inductive hypothesis.
Let $n \geq 10$ be some fixed number and assume that for all m such that $1 \leq m \leq n$ it holds that $m = 3a + 5b$ for some $a, b \in \mathbb{Z}$.
Induction.
Show $P(n+1)$; that is, $n+1 = 3a+5b$ for some $a, b \in \mathbb{Z}$. We know that we can write $n = 3a' + 5b'$ for some $a', b' \in \mathbb{Z}$ since $n \leq n$. So we have, $n+1 = (3a'+5b')+1$. We also have that $1 \leq n$, and for $a'' = -3$ and $b'' = 2$, $3a'' + 5b'' = 1$. Then $n+1 = (3a'+5b') + (3a''+5b'') = 3(a'+a'') + 5(b'+b'')$ where $a = a' + a''$ and $b = b' + b''$. Then for all $n \geq 8$ we have that $n = 3a+5b$ for some $a, b \in \mathbb{Z}$. □

Exercise 3. *Lewis, Zax: Exercise 4.5.*

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Solution.
Base cases.
$P(4) : a_4 = a_3 + a_2 + a_1 = 3 + 2 + 1 = 6$, and $6 < 16 = 2^4$.
$P(5) : a_5 = a_4 + a_3 + a_2 = 6 + 3 + 2 = 11$, and $11 < 32 = 2^5$.
$P(6) : a_6 = a_5 + a_4 + a_3 = 11 + 6 + 3 = 20$, and $20 < 64 = 2^6$.
Inductive hypothesis.
Let $n \geq 6$ be some fixed number and assume that for all m such that $1 \leq m \leq n$ it holds that $a_m < 2^m$.
Induction.
Show $P(n + 1)$; that is, $a_{n+1} < 2^{n+1}$.
We have $a_{n+1} = a_n + a_{n-1} + a_{n-2}$ and we know that $a_n < 2^n$, $a_{n-1} < 2^{n-1}$, and $a_{n-2} < 2^{n-2}$ since $n, n-1, n-2 \leq n$. Then $a_{n+1} < 2^n + 2^{n-1} + 2^{n-2}$. Factoring the right side of the inequality we get $a_{n+1} < 2^{n-2}(2^2 + 2 + 1) = 2^{n-2}(7)$. We have that $7 < 8 = 2^3$. Then $a_{n+1} < 2^{n-2}(2^3) = 2^{n+1}$, and for all $n \geq 4$, $a_n < 2^n$. □

Exercise 4. *Use strong induction to show that any positive integer greater than 1 can be written as a product of primes.*

Solution.
Base cases.
$P(2) = 2$.
Inductive hypothesis.
Let $n \geq 2$ be some fixed number and assume that for all m such that $1 \leq m \leq n$ it holds that n can be written as a product of primes.
Induction.
Show $P(n + 1)$; that is, $n + 1 = q_1 q_2 ... q_n$ where q_i is some prime and $1 \leq i \leq n$.
There are two cases: one in which $n+1$ is a prime (in which case we are done) or $n+1$ is composite. In this second case $n + 1$ can be written as a product $n + 1 = ab$, but both $a, b \leq n$ meaning they both can be written as a product of primes. Then $n + 1 = ab = q_1 q_2 ... q_n \cdot q_1' q_2' ... q_n'$, and $n + 1$ can be written as a product of primes. Then for any $n \in \mathbb{Z}$ where $n \geq 1$, n can be written as a product of primes.

Exercise 5. *Lewis, Zax: Exercise 8.1.*

Solution.
Define $\#_a(s)$ inductively.
Base case.
$\#_a(\lambda) = 0$.
Constructor cases.
Given $\#_a(s) = n$ where $s \in \Sigma^*$ and $x \in \Sigma$:
(C1) If $x = a$ then $\#_a(x \cdot s) = \#_a(s) + 1 = n + 1$.
(C2) If $x \neq a$ then $\#_a(x \cdot s) = \#_a(s) = n$.
Prove by structural induction that:
$\#_a(s \cdot t) = \#_a(s) + \#_a(t)$.
We define $P(n)$ as $\#_a(s \cdot n) = \#_a(s) + \#_a(n)$.
Base case.

$P(\lambda) : \#_a(s \cdot \lambda) = \#_a(s) = \#_a(s) + 0 = \#_a(s) + \#_a(\lambda)$.
Inductive Hypothesis.
Assume this is true up to t, that is $P(t) : \#_a(s \cdot t) = \#_a(s) + \#_a(t)$.
Induct.
Show this is true for $t \cdot u$, that is $P(t \cdot u)$, where $u \in \Sigma$. We want to show that $\#_a(s \cdot t \cdot u) = \#_a(s) + \#_a(t \cdot u)$.
If $u = a$:
By inductive definition: $\#_a(s \cdot t \cdot u) = \#_a(s \cdot t) + 1$.
By inductive hypothesis: $\#_a(s \cdot t \cdot u) = \#_a(s) + \#_a(t) + 1$.
By inductive definition: $\#_a(s \cdot t \cdot u) = \#_a(s) + \#_a(t \cdot u)$.

If $u \neq a$:
By inductive definition: $\#_a(s \cdot t \cdot u) = \#_a(s \cdot t) + 0$.
By inductive hypothesis: $\#_a(s \cdot t \cdot u) = \#_a(s) + \#_a(t) + 0$.
By inductive definition: $\#_a(s \cdot t \cdot u) = \#_a(s) + \#_a(t \cdot u)$.
Then we have that $P(t \cdot u)$ holds and that $\#_a(s \cdot t) = \#_a(s) + \#_a(t)$ is true for any strings s, t. □

Exercise 6. *Lewis, Zax: Exercise 8.4.*

Solution.
We will use structural induction to prove that $|u \cdot v| = |u| + |v|$.
We define $P(n)$ as $|u \cdot n| = |u| + |n|$.
Base case.
$P(\lambda) : |u \cdot \lambda| = |u| = |u| + 0 = |u| + |\lambda|$.
Inductive hypothesis.
Assume this is true up to t, that is $P(t)$ $|u \cdot t| = |u| + |t|$.
Induct.
We want to show this is true for $t \cdot a$, that $P(t \cdot a)$ holds, where $a \in \Sigma$, i.e. that $|u \cdot t \cdot a| = |u| + |t \cdot a|$.
By inductive definition: $|u \cdot t \cdot a| = |u \cdot t| + 1$.
By inductive hypothesis: $|u \cdot t \cdot a| = |u| + |t| + 1$.
By inductive definition: $|u \cdot t \cdot a| = |u| + |t \cdot a|$.
Then we have that $P(t \cdot a)$ holds and that $|u \cdot v| = |u| + |v|$ is true for any strings u, v. □

Exercise 7. *Lewis, Zax: Exercise 13.1.*

Solution.
$b \to c \to c \to d$.

Exercise 8. *Lewis, Zax: Exercise 13.3.*

Solution.
a.
$id(a) = 1$
$od(a) = 2$
$id(b) = 1$
$od(b) = 2$
$id(c) = 2$

$od(c) = 2$
$id(d) = 2$
$od(d) = 2$
$id(e) = 2$
$od(e) = 0$
b.
$a \to d \to b \to a.$
$a \to c \to d \to b \to a.$
$b \to c \to d.$
c.
$d(a, a) = 3$
$d(a, b) = 2$
$d(a, c) = 1$
$d(a, d) = 1$
$d(a, e) = 2$

$d(b, a) = 1$
$d(b, b) = 3$
$d(b, c) = 1$
$d(b, d) = 2$
$d(b, e) = 2$

$d(c, a) = 3$
$d(c, b) = 2$
$d(c, c) = 3$
$d(c, d) = 1$
$d(c, e) = 1$

$d(d, a) = 2$
$d(d, b) = 1$
$d(d, c) = 2$
$d(d, d) = 3$
$d(d, e) = 1$

$d(e, a) = \infty$
$d(e, b) = \infty$
$d(e, c) = \infty$
$d(e, d) = \infty$
$d(e, e) = \infty$