MA0301
ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2021

Set 5

Deadline: Wednesday 3 March, 2021, 23:59.

Exercise 1. *Show that for all integers $m > 0$*

$$\sum_{j=1}^{m} \frac{1}{j(j+2)} = \frac{m(3m+5)}{4(m+1)(m+2)}.$$

Solution. We will prove the statement by mathematical induction on m. The base case $m = 1$ is straightforward since $\frac{1}{1(1+2)} = \frac{1}{3} = \frac{1(3+5)}{4(1+1)(1+2)}$. For the inductive hypothesis, assume that the statement is true for some positive integer $m = k \geq 1$, i.e., we suppose that $\sum_{j=1}^{k} \frac{1}{j(j+2)} = \frac{k(3k+5)}{4(k+1)(k+2)}$. For the inductive step, we will prove that the statement is also true for $m = k+1$. Indeed, by splitting the sum and using the inductive hypothesis, we have that

$$\begin{aligned}
\sum_{j=1}^{k+1} \frac{1}{j(j+2)} &= \sum_{j=1}^{k} \frac{1}{j(j+2)} + \frac{1}{(k+!)(k+3)} \\
&= \frac{k(3k+5)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+3)} \\
&= \frac{k(3k+5)(k+1)(k+3) + 4(k+1)(k+2)}{4(k+1)^2(k+2)(k+3)} \\
&= \frac{k(3k+5)(k+3) + 4(k+2)}{4(k+1)(k+2)(k+3)} \\
&= \frac{(k+1)k(3k+5) + 2k(3k+5) + 4(k+1) + 4}{4(k+1)(k+2)(k+3)} \\
&= \frac{(k+1)(k(3k+5)+4) + 6k^2 + 10k + 4}{4(k+1)(k+2)(k+3)} \\
&= \frac{(k+1)(k(3k+5)+4) + (k+1)(6k+4)}{4(k+1)(k+2)(k+3)} \\
&= \frac{3k^2 + 11k + 8}{4(k+2)(k+3)} \\
&= \frac{(k+1)(3k+8)}{4(k+2)(k+3)} \\
&= \frac{(k+1)(3(k+1)+5)}{4((k+1)+1)((k+1)+2)}.
\end{aligned}$$

Hence the formula is true for $k+1$. By the mathematical induction principle, we conclude that the formula es true for any $m \in \mathbb{N}$. □

Exercise 2. *Show that for all non-negative integers m, we have that 3 divides the number $a_m = 2^{2m+1} + 1$.*

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Solution. By mathematical induction on m. For the base case $m = 1$, we have that $a_1 = 2^{2+1} + 1 = 9$ and clearly $3|a_1$. For the inductive hypothesis, assume that 3 divides to $a_k = 2^{2k+1} + 1$, for some positive integer $k \geq 1$. We will prove that $3|a_{k+1}$. Indeed, notice that

$$a_{k+1} = 2^{2k+3} + 1 = 2^2 \cdot 2^{2k+1} + 1 = (3+1)2^{2k+1} + 1 = 3 \cdot 2^{2k+1} + 2^{2k+1} + 1 = 3 \cdot 2^{2k+1} + a_k.$$

By induction hypothesis, we have that $3|a_k$. Since we clearly have that $3|3 \cdot 2^{2k+1}$, we can conclude that $3|(3 \cdot 2^{2k+1} + a_k)$ and hence $3|a_{k+1}$, as we wanted. By mathematical induction, we conclude that 3 divides a_m, for any $m \in \mathbb{N}$. □

Exercise 3. *Let $\{L_n\}$, $n \geq 0$ be the sequence of Lucas numbers, which are defined recursively, i.e., for $n > 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$. Show that for positive integers m*

$$\sum_{i=1}^{m} iL_i = mL_{m+2} - L_{m+3} + 4.$$

Solution. By induction on m. For the base case $m = 1$, we have $1 \cdot \ell_1 = 1$ and $1 \cdot \ell_3 - \ell_4 + 4 = 4 - 7 + 4 = 1$. Now, assume that the formula holds for a positive integer $k \geq 1$. We will prove the result for $k + 1$. By splitting the sum and using the inductive hypothesis, we have

$$\begin{aligned}
\sum_{i=1}^{k+1} i\ell_i &= \sum_{i=1}^{k} i\ell_i + (k+1)\ell_{k+1} \\
&= k\ell_{k+2} - \ell_{k+3} + 4 + (k+1)\ell_{k+1} \\
&= k(\ell_{k+2} + \ell_{k+1}) + \ell_{k+1} - \ell_{k+3} + 4 \\
&= (k-1)\ell_{k+3} + \ell_{k+1} + 4 + 2\ell_{k+3} - 2\ell_{k+3} \\
&= (k+1)\ell_{k+3} + \ell_{k+1} - \ell_{k+3} - \ell_{k+3} + 4 \\
&= (k+1)\ell_{k+3} + \ell_{k+1} - \ell_{k+1} - \ell_{k+2} - \ell_{k+3} + 4 \\
&= (k+1)\ell_{k+3} - \ell_{k+2} - \ell_{k+3} + 4 \\
&= (k+1)\ell_{k+3} - \ell_{k+4} + 4,
\end{aligned}$$

where we used the definition of Lucas' numbers. Hence the formula holds for $k + 1$. By mathematical induction, we have that $\sum_{i=1}^{m} i\ell_i = m\ell_{m+2} - \ell_{m+3} + 4$ for any $m \geq 1$.

□

Exercise 4. *Show that for all integers $m > 0$*

$$\sum_{i=1}^{m} (-1)^{i+1} i^2 = (-1)^{m+1} \sum_{i=1}^{m} i.$$

Solution. By induction on m. For the base case $m = 1$, observe that $(-1)^2 1^2 = 1 = (-1)^2 1$. Now, for the inductive hypothesis, assume that the formula holds for a positive integer $k \geq 1$. We will prove the result for $k+1$. Indeed, by splitting the sum, using the inductive hypothesis and Gauss' Formula, we have

$$\begin{aligned}\sum_{i=1}^{k+1}(-1)^{i+1}i^2 &= \sum_{i=1}^{k}(-1)^{i+1}i^2 + (-1)^{k+2}(k+1)^2\\ &= (-1)^{k+1}\sum_{i=1}^{k} i + (-1)^{k+2}(k+1)^2\\ &= (-1)^{k+1}\frac{k(k+1)}{2} + (-1)^{k+2}(k+1)^2\\ &= (-1)^{k+2}\frac{2(k+1)^2 - k(k+1)}{2}\\ &= (-1)^{k+2}\frac{k^2+3k+2}{2}\\ &= (-1)^{k+2}\frac{(k+1)(k+2)}{2}\\ &= (-1)^{k+2}\sum_{i=1}^{k+1} i.\end{aligned}$$

Hence the formula holds for $k+1$. By mathematical induction, we conclude that the formula holds for any $m \geq 1$. □

Exercise 5. *For each of the following relations, determine whether the relation is reflexive, symmetric, antisymmetric, or transitive:*

(1) $R \subseteq \mathbb{N} \times \mathbb{N}$ *where* $(a, b) \in R$ *if* a *divides* b*,*
(2) *For given a universe* $\mathcal{U}$ *and a fixed subset* C *of* $\mathcal{U}$*, define* R *on* $\mathcal{P}(\mathcal{U})$ *as follows: For* $A, B \subseteq \mathcal{U}$ *we have* $(A, B) \in R$ *if* $A \cap C = B \cap C$*.*
(3) *On the set* A *of all lines in* R^2*, define the relation* R *for two lines* $l_!$ *and* l_2 *by* $(l_1, l_2) \in R$ *if* l_1 *is perpendicular to* l_2*.*
(4) R *is the relation on* $\mathbb{Z}$ *where* $(x, y) \in R$ *if* $x + y$ *is odd.*

Solution. (1) The relation is reflexive since $a|a$ for any $a \in \mathbb{N}$. The relation is not symmetric, since $1|2$ but $2 \not| 1$. It is antisymmetric: if $a|b$ and $b|a$, we have that there exist $x, y \in \mathbb{N}$ such that $ax = b$ and $by = a$. Both equalities imply that $b = ax = byx$. This implies that $xy = 1 \Leftrightarrow x = y = 1$. Hence $a = b$. Finally, the relation is transitive: if $a|b$ and $b|c$, then there exist $x, y \in \mathbb{N}$ such that $ax = b$ and $by = c$. Both equalities imply that $c = by = (ax)y = a(xy)$. Since $xy \in \mathbb{N}$, the equation $c = a(xy)$ implies that $a|c$. Hence R is transitive.

(2) The relation is reflexive: If A is a subset of $\mathcal{U}$, we clearly have that $A \cap C = A \cap C$. The relation is symmetric: if $(A, B) \in R$, then $A \cap C = B \cap C$. This implies that $B \cap C = A \cap C$ and then $(B, A) \in R$. The relation is also transitive: if $(A, B), (B, D) \in R$, then $A \cap C = B \cap C$ and $\cap C = D \cap C$. Both equalities imply that $A \cap C = D \cap C$ and then $(A, D) \in R$. The relation is not anti-symmetric: if $\mathcal{U} = \{1, 2, 3\}, C = \{3\}, A = \{1\}$, and $B = \{2\}$, we have that $A \cap C = \emptyset = B \cap C$ but $A \neq B$.

(3) The relation is not reflexive since a line is not perpendicular to itself. It is clearly symmetric. It is not transitive: the lines $x = 0$ and $y = 0$ are perpendicular, $y = 0$ and $x = 1$ are perpendicular, but $x = 0$ and $x = 1$ are not perpendicular. It is not anti-symmetric: $x = 0$ and $y = 0$ are perpendicular but the lines are not the same.

(4) It is not reflexive, since $1 \in \mathbb{Z}$ and $(1, 1) \notin R$ since $1 + 1 = 2$ is not odd. It is clearly symmetric: if $(x, y) \in R$ then $x + y$ is odd. This implies that $y + x$ is odd and hence $(y, x) \in R$. It is not transitive: we have that $(1, 2), (2, 3) \in R$ since $1 + 2 = 3$ and $2 + 3 = 5$ but $1 + 3 = 4$ is not odd. Finally, it is not anti-symmetric since $(1, 2), (2, 1) \in R$ since $1 + 2 = 3$ but $1 \neq 2$.

□

Exercise 6. *Define $R \subseteq \mathbb{N}^2 \times \mathbb{N}^2$ the relation $((a,b),(c,d)) \in R \Leftrightarrow ad = bc$. Show that R is an equivalence relation.*

Solution. We will show that R is an equivalence relation.

- Reflexivity. Let $(a,b) \in \mathbb{N}^2$. We clearly have that $ab = ba$, so $((a,b),(a,b)) \in R$.
- Symmetry. Observe that $((a,b),(c,d)) \in R \Leftrightarrow ad = bc \Leftrightarrow cb = da \Leftrightarrow ((c,d),(a,b)) \in R$.
- Transitivity. Assume that $((a,b),(c,d)),((c,d),(e,f)) \in R$. Then we have $ad = bc$ and $cf = de$. Multiplying the last equation by a, we have $cfa = dea = ade$. Since $ad = bc$, we have $cfa = (ad)e = (bc)e$. Dividing by c, we get $af = be$. Hence $((a,b),(e,f)) \in R$.

We conclude that R is an equivalence relation. □

Exercise 7. *Let $A = \{1,2,3,4,5\} \times \{1,2,3,4,5\}$. Define the relation R on A by $((x,y),(u,v)) \in R$ if $x + y = u + v$. Show that R is an equivalence relation on A and determine the equivalence classes $[(1,3)]$, $[(2,4)]$ and $[(1,1)]$.*

Solution. We will show that R is an equivalence relation.

- Reflexivity: If $(a,b) \in A$, then clearly have that $a+b = a+b$ and hence $((a,b),(a,b)) \in R$.
- Symmetry: If $((a,b),(u,v)) \in R$ then $a+b = u+v$. This implies that $u+v = a+b$ and hence $((u,v),(a,b)) \in R$.
- Transitivity: If $((a,b),(u,v)),((u,v),(x,y)) \in R$ then we have that $a+b = u+v$ and $u+v = x+y$. Both equations imply that $a+b = x+y$ and hence $((a,b),(x,y)) \in R$.

Hence R is an equivalence relation. Now, recall that

$$[(a,b)] = \{(u,v) \in A \ : \ ((a,b),(u,v)) \in R\}.$$

For the equivalence class of $(1,3)$, we have to find all the pairs $(u,v) \in A$ such that $u+v = 1+3 = 4$. It is easy to see that

$$[(1,3)] = \{(1,3),(3,1),(2,2)\}.$$

In a similar way, we have

$$\begin{aligned} [(2,4)] &= \{(1,5),(5,1),(2,4),(4,2),(3,3)\}, \\ [(1,1)] &= \{(1,1)\}. \end{aligned}$$

□

Exercise 8. *If $A = \{1,2,3,4,5,6,7\}$, define the relation R on A by $(x,y) \in R$ if $x-y$ is multiple of 3. Show that R is an equivalence relation on A and determine the equivalence classes and partition of A induced by R.*

Solution. We will show that R is an equivalence relation.

- Reflexivity: If $x \in A$, then clearly have that 3 divides to $0 = x - x$ and hence $(x,x) \in R$.
- Symmetry: If $(x,y) \in R$ then 3 divides $x-y$. Since $y - x = -(x-y)$, then we also have that 3 divides $y - x$. Hence $(y,x) \in R$.
- Transitivity: If $(x,y),(y,z) \in R$, we have that 3 divides to $x-y$ and $y-z$. This implies that 3 divides the sum of the numbers $x-y$ and $y-z$ that is equal to $x-y+y-z = x-z$. Hence 3 divides to $x-z$, i.e. $(x,z) \in R$.

Hence R is an equivalence relation. We find now the equivalence classes:

- For the equivalence class of 1, we have to find all the elements $x \in A$ such that 3 divides $1-x$. A quick inspection produces that x can be $1,4,7$.
- For the equivalence class of 2, we have to find all the elements $x \in A$ such that 3 divides $2-x$. A quick inspection produces that x can be $2,5$.

- For the equivalence class of 3, we have to find all the elements $x \in A$ such that 3 divides $3 - x$. A quick inspection produces that x can be $3, 6$.

Hence the equivalence classes of R are

$$[1] = \{1, 4, 7\},\ [2] = \{2, 5\},\ [3] = \{3, 6\}$$

and the partition of A induced by R is $\{\{1, 4, 7\}, \{2, 5\}, \{3, 6\}\}$. $\square$