MA0301
ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2021

Solutions Set 10

Exercise 1. *For the graph in figure 11.7, determine:*
a) a walk from b to d that is not a trail.
Solution. *An $x-y$ walk to is a sequence of alternating edges and vertices that start at vertex x and end at vertex y. A trail is a walk that has no repeating edges, then to determine a $b-d$ walk that is not a trail we want a sequence that begins at b, ends at d, and repeats one or more edges:*
$b \to a \to c \to b \to a \to c \to d$.

b) a $b-d$ trail that is not a path.
Solution. *A trail is a walk that has no repeating edges and a path is a walk that has no repeating vertices, then to determine a trail that is not a path we will repeat one or more vertices but not repeat any edges:*
$b \to a \to c \to b \to e \to d$

c) a path from b to d.
Solution. *A path is a walk with no repeating vertices:*
$b \to e \to d$.

d) a closed walk from b to b that is not a circuit.
Solution. *A circuit is a closed trail, so we will write a closed walk which repeats one or more edges:*
$b \to e \to f \to g \to e \to f \to g \to e \to b$

e) a circuit from b to b that is not a cycle.
Solution. *A circuit is a closed trail and a cycle is a closed path, then we will determine a path that has one or more repeating vertices but no repeating edges:*
$b \to e \to f \to g \to e \to b$.

f) a cycle from b to b.
Solution. *We will determine a closed path (no repeating vertices):*
$b \to a \to c \to b$.

Date: April 12, 2021.

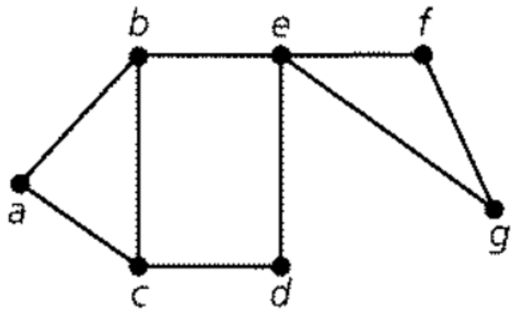

Figure 11.7

Note: there are many correct solutions and several ways to write these solutions.

Exercise 2. *Give an example of a connected graph where removing any edge of G results in a disconnected graph.*

Solution.
A connected graph, G, is one in which there is a path between any two distinct vertices. We can start with that and then look at different scenarios in which the removal of any edge from G also removes the path between some set of two distinct vertices:

a •——• b •——• c

Note: there are many correct solutions.

Exercise 3. *Figure 11.10 shows an undirected graph representing a section of a department store. The vertices indicated where cashiers are located; the edges denote unblocked aisles between cashiers. The department store wants to set up a security system where guards are placed at certain cashier locations so that each cashier either has a guard at his or her location or is only one aisle away from a cashier who has a guard. What is the smallest number of guards needed?*

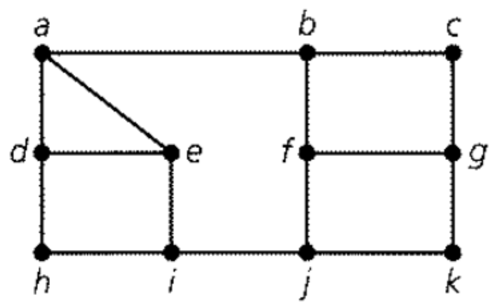

Figure 11.10

Solution. 3. We can do this by placing a guard with cashier d, j, and c.

Exercise 4. *For each pair of graphs in figure 11.29, determine whether or not the graphs are isomorphic.*

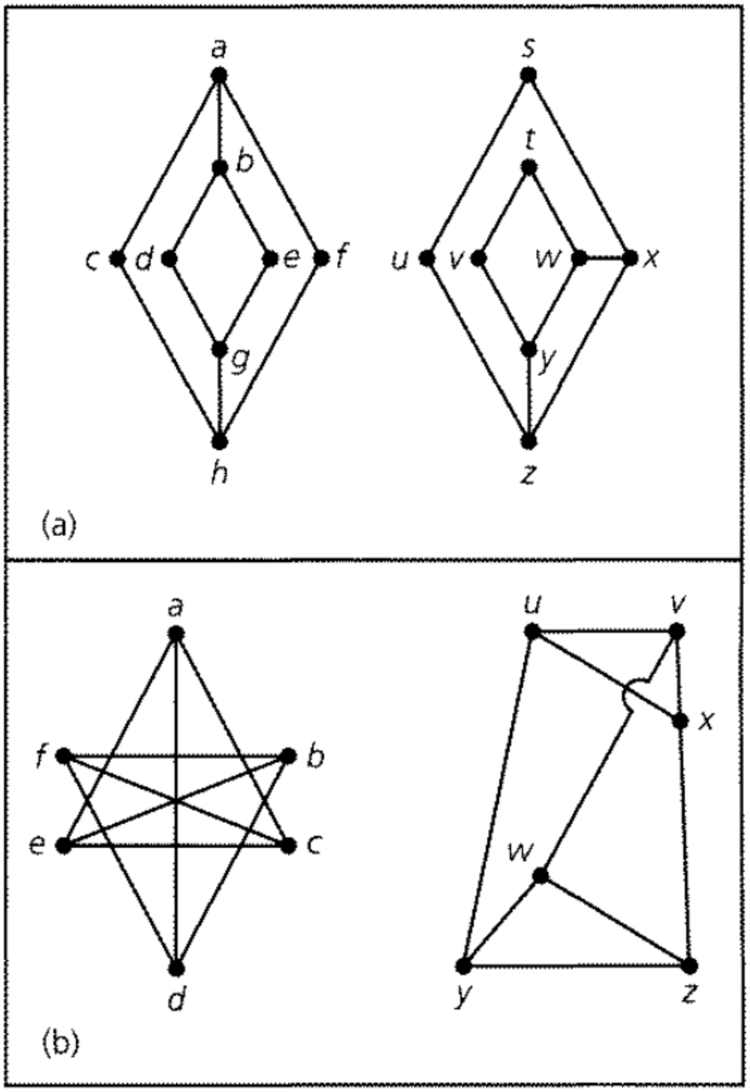

Figure 11.29

Solution. (a) Not isomorphic. Initially we can see that the two graphs have an important difference in edge placement, even though there are the same number of edges, because of this we can look for a contradiction which will show that the two graphs are not isomorphic. In order to create a one-to-one correspondence we need to map vertices from the first graph onto vertices in second graph which have the same number of adjacent vertices. For example vertex a would need to be mapped onto vertex x, w, y, or z. Lets assume we map onto x (though the following argument is relevant regardless of which vertex we choose to map on to). There exists a cycle of length 4 ($x \to w \to y \to z \to x$) for which there is no correspondence in the first graph (the shortest cycle from a to a is length 5). Then the two graphs can not be isomorphic.

(b) Here it is more difficult to tell if the two graphs are isomorphic. We will likely need to try a few different mappings and check the edge correspondences to see if we can find something appropriate.

$f(a) = u$
$f(c) = v$
$f(e) = x$
$f(d) = y$
$f(f) = w$
$f(b) = z$
This is a one-to-one and onto mapping, so the first part of our definition is fulfilled. Now we will check the edges. Both graphs have 9 edges:
$\{a, e\} = \{f(a), f(e)\} = \{u, x\}$
$\{e, c\} = \{f(e), f(c)\} = \{x, v\}$
$\{c, a\} = \{f(c), f(a)\} = \{v, u\}$
$\{f, b\} = \{f(f), f(b)\} = \{w, z\}$
$\{b, d\} = \{f(b), f(d)\} = \{z, y\}$
$\{d, f\} = \{f(d), f(f)\} = \{y, w\}$
$\{e, b\} = \{f(e), f(b)\} = \{x, z\}$
$\{f, c\} = \{f(f), f(c)\} = \{w, v\}$
$\{a, d\} = \{f(a), f(d)\} = \{u, y\}$
Then we also have edge correspondence and the two graphs are isomorphic.

Exercise 5. *a) Find an Euler circuit for the graph in figure 11.44.*
Solution. *An Euler circuit is a closed trail which uses all the edges in the graph. There is some trial and error here.*
$b \to c \to g \to k \to j \to i \to h \to d \to a \to b \to d \to e \to b \to g \to j \to f \to i \to e \to f \to b.$

b) If the edge $\{d, e\}$ is removed from this graph find an Euler trail for the resulting subgraph.
Solution. *An Euler trail is an open trail which traverses all edges in the graph. Again trial and error.*
$d \to a \to b \to c \to g \to k \to j \to i \to h \to d \to b \to e \to f \to b \to g \to j \to f \to i \to e$

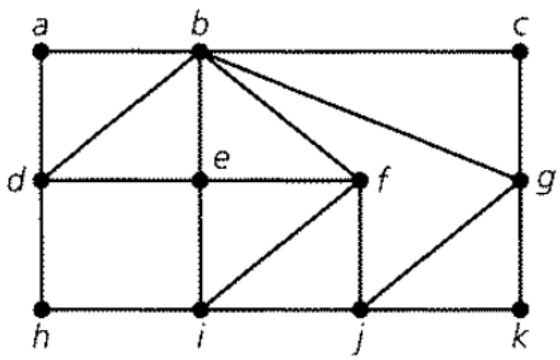

Figure 11.44

Exercise 6. *a) Let $G = (V, E)$ be an undirected graph, with $G_1 = (V_1, E_1)$ a subgraph of G. Under what condition(s) is G_1 not an induced subgraph of G?*

Solution. *G_1 is not an induced subgraph of G if E_1 does not contain all edges from E which correspond to the vertices in U, the nonempty subset of V.*

b) For the graph G in figure 11.27a, find a subgraph that is not an induced subgraph.

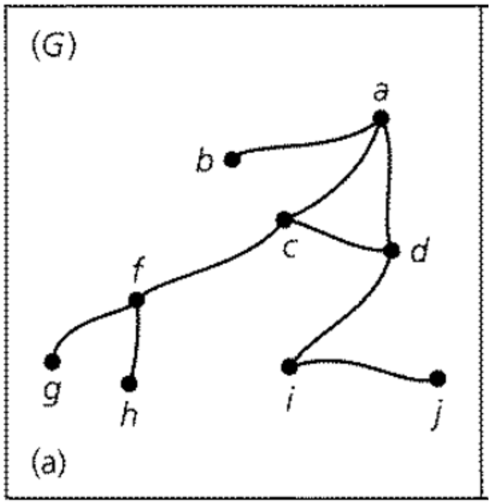

Figure 11.27

Solution. *$G = (V, E_1)$ where $V = \{a, b, c, d\}$ and $E_1 = \{\{b, a\}, \{a, c\}, \{a, d\}\}$ is not an induced subgraph because $\{c, d\} \notin E_1$.*

Exercise 7. *If $G = (V, E)$ is a connected graph with $|E| = 17$ and $deg(v) \geq 3$ for all $v \in V$. What is the maximum value for $|V|$?*

Solution. We will use the handshaking lemma to determine the maximum value of $|V|$: $\sum_{v \in V} deg(v) = 2|E| = 2 \times 17 = 34$. To maximize the number of vertices we want to minimize the degree of each vertex. We know that at a minimum $deg(v) = 3$. Then $\frac{34}{3} = 11 + \frac{1}{3}$, and 11 is the maximum of $|V|$.