MA0301
ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2020

Set 3

You can deliver the exercises (before 10:30am on the indicated day) on the 3 floor of the central building, in an area called Matteland. The box number is 0613.
If you want feedback on each exercise, write "want comments"/"nsker retting", otherwise your TA will only write "godkjent"/"ikke godkjent".

Exercise 1. *Use the rules of inference to show that the trueness of:*

$$i)\,(r \vee p) \wedge (\neg q \vee \neg p) \Rightarrow (r \vee \neg q)$$

$$ii)\,(z \vee \neg s) \wedge (p \wedge \neg q) \wedge (\neg z \vee r) \wedge (\neg(p \wedge \neg q) \vee s) \Rightarrow r$$

Solution. 1)

	Step	**Reason**
1	$\neg q \vee \neg p$	Premise
2	$q \Rightarrow \neg p$	Equivalence of (1)
3	$r \vee p$	Premise
4	$\neg(\neg p) \vee r$	Double negation and commutativity in (3)
5	$\neg p \Rightarrow r$	Equivalence of (4)
6	$q \Rightarrow r$	Law of Syllogism from (2) and (5)
7	$\neg q \vee r$	Equivalence of (6)
8	$r \vee \neg q$	Commutativity in (8)

2)

	Step	**Reason**
1	$p \wedge \neg q$	Premise
2	$\neg(p \wedge \neg q) \vee s$	Premise
3	$(p \wedge \neg q) \Rightarrow s$	Equivalence of (2)
4	s	Modus Ponens from (1) and (3)
5	$z \vee \neg s$	Premise
6	$s \Rightarrow z$	Equivalence of (5)
7	z	Modus Ponens from (4) and (6)
8	$\neg z \vee r$	Premise
9	$z \Rightarrow r$	Equivalence of (8)
10	r	Modus Ponens from (7) and (9)

□

Date: 4 February, 2020.

Exercise 2. *Express the following statements in English. Then determine which of them are true in the natural numbers,* $\mathbb{N} := \{0, 1, 2, 3, 4, 5, \ldots\}$ *and which are true in real numbers* $\mathbb{R}$.

a) $\forall x\ \forall y\ ((x > y) \Rightarrow \exists z\ ((x > z) \wedge (z > y)))$

b) $\forall x\ (x = 0 \vee \neg(x + x = x))$

Solution. a) *If x and y are such that x is greater than y, then there exists z such that x is greater than z and z is greater than y.*

This is not true in the natural numbers. For instance, take $x = 1$ and $y = 0$. Then $x > y$ but there is no natural number z between 0 and 1. The statement is true in the real numbers because of the density of real numbers (there is a real number between two real numbers).

2) *For any number x, we have that x is equal to zero or the double of x is different than x.*

This is true in both $\mathbb{N}$ and $\mathbb{R}$. The following argument works in both cases. If x is not zero and $2x = x$, we can divide by x (since $x \neq 0$) and then we would have $2 = 1$, which is a contradiction. Hence, we cannot have that $2x = x$. □

Exercise 3. *What is the power set of* $A := \{\{a, b\}, \{c\}, \{d, e, f\}\}$*?*

Solution. The power set of A is defined as the collection of subsets of A. We have then that the power set of A is the following:

$$\mathcal{P}(A) = \{\emptyset, \{\{a, b\}\}, \{\{c\}\}, \{\{d, e, f\}\}, \{\{a, b\}, \{c\}\}, \{\{c\}, \{d, e, f\}\}, \{\{a, b\}, \{d, e, f\}\}, A\}$$

□

Exercise 4. *Let X and Y be two sets. Show that* $X - Y = X \cap \overline{Y}$.

Solution. Assume that X and Y are sets in the universe $\mathcal{U}$. Then

$$\begin{aligned}
x \in X - Y \quad &\Leftrightarrow \quad x \in X \wedge x \notin Y \\
&\Leftrightarrow \quad x \in X \wedge x \in \mathcal{U} \wedge x \notin Y \qquad \text{(identity law } A \cap \mathcal{U} = A\text{)} \\
&\Leftrightarrow \quad x \in X \wedge x \in \overline{Y} \\
&\Leftrightarrow \quad x \in X \cap \overline{Y}.
\end{aligned}$$

Hence $X - Y = X \cap \overline{Y}$. □

Exercise 5. *For two sets X and Y, show that $A := X - Y$ and $B := X \cap Y$ are disjoint sets and that $X = A \cup B$.*

Solution. Using Exercise 4, we have that $A = X - Y = X \cap \overline{Y}$. Hence

$$A \cap B = (X - Y) \cap (X \cap Y) = (X \cap \overline{Y}) \cap (X \cap Y) = X \cap X \cap (Y \cap \overline{Y}) = X \cap \emptyset = \emptyset,$$

by using the Commutativity Law, that $Y \cap \overline{Y} = \emptyset$ and the Absorption Law. Hence A and B are disjoint sets. In a similar way, we have

$$X \cup Y = (X \cap \overline{Y}) \cup (X \cap Y) = X \cup (Y \cap \overline{Y}) = X \cup \emptyset = X,$$

where in the second equality we use the Distributive Law and the Absorption Law in the last equality. □

Exercise 6. *Let A, B, C be sets. The symmetric difference was defined by $A \triangle B := (A - B) \cup (B - A)$. Show that a) $A \triangle B = (A \cup B) - (A \cap B)$, b) $A \triangle B = B \triangle A$ and that c) $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.*

Solution. 1) Using Exercise 4 and Distributive Law, we have that

$$\begin{aligned}
A \triangle B &= (A - B) \cup (B - A) \\
&= (A \cap \overline{B}) \cup (B \cap \overline{A}) \\
&= ((A \cap \overline{B}) \cup B) \cap ((A \cap \overline{B}) \cup \overline{A}) \\
&= ((A \cup B) \cap (\overline{B} \cup B)) \cap ((A \cup \overline{A}) \cap (\overline{B} \cup \overline{A})) \\
&= ((A \cup B) \cap \mathcal{U}) \cap (\mathcal{U} \cap (\overline{B} \cup \overline{A})) \\
&= (A \cup B) \cap (\overline{A \cap B}) \qquad \text{(Identity Law and DeMorgan's Law)} \\
&= (A \cup B) - (A \cap B)) \qquad \text{(Exercise 4).}
\end{aligned}$$

2) This immediately follows from Commutativity Law of union of sets:

$$A \triangle B = (A - B) \cup (B - A) = (B - A) \cup (A - B) = B \triangle A.$$

3)

$$\begin{aligned}
A \triangle (B \triangle C) &= (A \cap (\overline{B \triangle C})) \cup ((B \triangle C) \cap \overline{A}) && \text{(Definition of } \triangle \text{ a} \\
&= (A \cap (\overline{(B \cap \overline{C}) \cup (C \cap \overline{B})})) \cup (((B \cap \overline{C}) \cup (C \cap \overline{B})) \cap \overline{A}) && \text{(Definition of } \triangle \text{ a} \\
&= (A \cap ((\overline{B} \cup C) \cap (\overline{C} \cup B))) \cup (((B \cap \overline{C}) \cup (C \cap \overline{B})) \cap \overline{A}) && \text{DeMorgan's Law} \\
&= (A \cap ((\overline{B} \cap (\overline{C} \cup B)) \cup (C \cap (\overline{C} \cup B))) \cup (((B \cap \overline{C}) \cup (C \cap \overline{B})) \cap \overline{A}) && \text{Distributive Law} \\
&= (A \cap ((\overline{B} \cap \overline{C}) \cup (\overline{B} \cap B)) \cup (C \cap \overline{C}) \cup (C \cap B))) \cup (((B \cap \overline{C}) \cup (C \cap \overline{B})) \cap \overline{A}) && \text{Distributive Law} \\
&= (A \cap ((\overline{B} \cap \overline{C}) \cup \emptyset \cup \emptyset \cup (C \cap B))) \cup (((B \cap \overline{C}) \cup (C \cap \overline{B})) \cap \overline{A}) && \text{Disjoint intersect} \\
&= (A \cap ((\overline{B} \cap \overline{C}) \cup (C \cap B))) \cup (((B \cap \overline{C}) \cup (C \cap \overline{B})) \cap \overline{A}) && \text{Identity Law} \\
&= (A \cap \overline{B} \cap \overline{C}) \cup (A \cap C \cap B) \cup (B \cap \overline{C} \cap \overline{A}) \cup (C \cap \overline{B} \cap \overline{A}) && \text{Distributive Law}
\end{aligned}$$

Now, by part 2, we have that $(A \triangle B) \triangle C = C \triangle (A \triangle B)$. We can use the previous development but just changing the label of the sets:

$$C \triangle (A \triangle B) = (C \cap \overline{A} \cap \overline{B}) \cup (C \cap B \cap A) \cup (A \cap \overline{B} \cap \overline{C}) \cup (B \cap \overline{A} \cap \overline{C}).$$

Finally, because union and intersection of sets is commutative, we conclude that

$$\begin{aligned}
A \triangle (B \triangle C) &= (A \cap \overline{B} \cap \overline{C}) \cup (A \cap C \cap B) \cup (B \cap \overline{C} \cap \overline{A}) \cup (C \cap \overline{B} \cap \overline{A}) \\
&= (A \cap \overline{B} \cap \overline{C}) \cup (C \cap B \cap A) \cup (B \cap \overline{A} \cap \overline{C}) \cup (C \cap \overline{A} \cap \overline{B}) \quad \text{(rearranging intersections)} \\
&= (C \cap \overline{A} \cap \overline{B}) \cup (C \cap B \cap A) \cup (A \cap \overline{B} \cap \overline{C}) \cup (B \cap \overline{A} \cap \overline{C}) \quad \text{(rearranging unions)} \\
&= C \triangle (A \triangle B) = (A \triangle B) \triangle C,
\end{aligned}$$

that is what we wanted to prove. □

Exercise 7. *Let $A := \{1, 2, 3, 4, 5, 6, \{1\}\}$. Determine the set B such that $B = A \cup (A \cap \mathcal{P}(A))$.*

Solution. One can try to compute $\mathcal{P}(A)$ and the respective intersection and union with A. However, we can proceed in general as follows: note that $A \cap \mathcal{P}(A) \subset A$ and by Absorption law, $B = A \cup (A \cap \mathcal{P}(A)) = A$. □

Exercise 8. *For two sets X and Y to show that they are equal, i.e., $X = Y$, you learned that you must show that $X \subseteq Y$ and $Y \subseteq X$. Show that this is equivalent to showing: if $x \in X$ then $x \in Y$ and if $x \notin X$ then $x \notin Y$. Now, use the latter to show that*

$$X \times (Y \cup Z) = (X \times Y) \cup (X \times Z).$$

Solution. We know that $X \subseteq Y$ is equivalent to the statement $x \in X \Rightarrow x \in Y$. We also have that $Y \subseteq X$ is equivalent to $x \in Y \Rightarrow x \in X$. In general, we know that for $(p \Rightarrow q) \equiv (\neg q \Rightarrow \neg p)$. So, $(x \in Y \Rightarrow x \in X) \Leftrightarrow (x \notin X \Rightarrow x \notin Y)$. Hence $X \subseteq Y \wedge Y \subseteq X$ is equivalent to the statement $x \in X \Rightarrow x \in Y$ and $x \notin X \Rightarrow x \notin Y$.

Now, assume that $(a, b) \in X \times (Y \cup Z)$. This is equivalent to say that $a \in X$ and $b \in Y \cup Z$. The latter statement implies (it is equivalent, actually) that $b \in Y \vee b \in Z$. Hence

$$(a, b) \in X \times (Y \cup Z) \Rightarrow a \in X \wedge (b \in Y \vee b \in Z) \Rightarrow (a \in X \wedge b \in Y) \vee (a \in X \wedge b \in Z)$$

$$\Rightarrow ((a, b) \in X \times Y) \vee ((a, b) \in (X, Z)) \Rightarrow (a, b) \in (X \times Y) \cup (X \times Z).$$

We have proven that $(a, b) \in X \times (Y \cup Z)$ implies that $(a, b) \in (X \times Y) \cup (X \times Z)$. Now, assume that $(a, b) \notin X \times (Y \cup Z)$. This implies that $a \notin X$ or $b \notin Y \cup Z$. The second statement implies that $b \notin Y$ and $b \notin Z$. So we have that

$$a \notin X \vee (b \notin Y \wedge b \notin Z) \Rightarrow (a \notin X \vee b \notin Y) \wedge (a \notin X \vee b \notin Z)$$

$$\Rightarrow ((a, b) \notin X \times Y) \wedge ((a, b) \notin X \times Z)) \Rightarrow (a, b) \notin (X \times Y) \cup (X \times Z).$$

So, he have proven that $(a, b) \notin X \times (Y \cup Z)$ implies that $(a, b) \notin (X \times Y) \cup (X \times Z)$. This allows to conclude that $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$. □

Exercise 9. *For two sets X and Y, prove that the following three statements are equivalent:*
i) $X \subseteq Y$, ii) $X \cap Y = X$, iii) $X \cup Y = Y$.

Solution. We have to prove $i) \Leftrightarrow ii), ii) \Leftrightarrow iii)$ and $iii) \Leftrightarrow i)$. By Law of Syllogism, this is equivalent to show $i) \Rightarrow ii), ii) \Rightarrow iii)$ and $iii) \Rightarrow i)$.

Proof of $i) \Rightarrow ii)$. Assume that $X \subseteq Y$. Then $ii)$ follows by the Absorption Law. We can prove it as follows: by definition of intersection, we have that $X \cap Y \subseteq X$. On the other hand, consider $x \in X$. By $i)$, we have that $x \in Y$. By conjunction, we have that $x \in X$ and $x \in Y$, i.e., $x \in X \cap Y$. We have shown that $X \subseteq X \cap Y$. Hence $X = X \cap Y$.

Proof of $ii) \Rightarrow iii)$. Assume that $X \cap Y = X$. Then

$$X \cup Y = (X \cap Y) \cup Y = (X \cup Y) \cap (Y \cup Y) = (X \cup Y) \cap Y = Y,$$

where we used the Absorption Law (or the first implication that we proved) in the last equality since $Y \subseteq X \cup Y$.

Proof of $iii) \Rightarrow i)$. Assume now that $X \cup Y = Y$. In general, we know that $X \subseteq X \cup Y$, but since $X \cup Y = Y$, we can conclude that $X \subseteq Y$.

□

Exercise 10. *For two sets X and Y, show that i) $(X \cup Y) \cap (X \cup \overline{Y}) = X$, ii) $(X \cap Y) \cup (X \cap \overline{Y}) = X$.*

Solution. i) By Distributive Law, we have

$$(X \cup Y) \cap (X \cup \overline{Y}) = X \cup (Y \cap \overline{Y}) = X \cup \emptyset = X,$$

since $Y \cap \overline{Y} = \emptyset$.

ii) Also by Distributive law

$$(X \cap Y) \cup (X \cap \overline{Y}) = X \cap (Y \cup \overline{Y}) = X \cap \mathcal{U} = X,$$

since $Y \cup \overline{Y} = \mathcal{U}$. $\square$