MA0301
ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2020

Solutions Set 1

Exercise 1. *a) Write down the truth table for* $p \wedge (\neg p \wedge q)$.

b) Write down the truth tables for $p \wedge (q \vee r)$ *and* $(p \wedge q) \vee (p \wedge r)$. *What is your conclusion? What can you say about the two compound propositions* $p \vee (q \wedge r)$ *and* $(p \vee q) \wedge (p \vee r)$*?*

Solution. a)

p	q	$\neg p$	$\neg p \wedge q$	$p \wedge (\neg p \wedge q)$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

Hence the proposition $p \wedge (\neg p \wedge q)$ is unsatisfiable (one can also verify by associativity $p \wedge (\neg p \wedge q) \equiv (p \wedge \neg p) \wedge q$ and by noticing that $p \wedge \neg p$ is always unsatisfiable).

b)

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Since the fifth and eighth column are identical, we conclude that $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ are equivalent. We can also show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are equivalent by writing down the truth tables as in the latter case, since fifth and eighth column are identical in the following table:

Date: January 6, 2020.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

□

Exercise 2. *Use truth tables to:*

a) Verify that $\neg(p \wedge q)$ *and* $\neg p \vee \neg q$ *are logically equivalent.*

b) Verify that $p \vee (q \vee r)$ *and* $(p \vee q) \wedge r$ *are not logically equivalent.*

Solution. a)

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Since fourth and seventh columns are identical, we conclude that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

b) Consider the following assignments of truth values for p, q and r.

p	q	r	$q \vee r$	$p \vee (q \vee r)$	$p \vee q$	$(p \vee q) \wedge r$
T	T	F	T	T	T	F

Hence, the truth tables for $p \vee (q \vee r)$ and $(p \vee q) \wedge r$ cannot be the same. We have that the two propositions are not equivalent. □

Exercise 3. *1) Write down the truth table of the so-called EXCLUSIVE OR:* $p \oplus q$*, which is defined to be true if either p is true and q is false, or p is false and q is true, and it is false in all other cases.*

2) Verify that $p \oplus q$ *is logically equivalent to* $(p \wedge \neg q) \vee (\neg p \wedge q)$*.*

Solution. 1) According to the definition $p \oplus q$ is true if and only if p and q have different truth values. The truth table is the following:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

2) We proceed to write down the table truth for $p \oplus q$ and $(p \wedge \neg q) \vee (\neg p \wedge q)$:

p	q	$p \oplus q$	$\neg p$	$\neg q$	$p \wedge \neg q$	$\neg p \wedge q$	$(p \wedge \neg q) \vee (\neg p \wedge q)$
T	T	F	F	F	F	F	F
T	F	T	F	T	T	F	T
F	T	T	T	F	F	T	T
F	F	F	T	T	F	F	F

Third and eighth columns are identical, then $p \oplus q$ is logically equivalent to $(p \wedge \neg q) \vee (\neg p \wedge q)$. □

Exercise 4. *Write down the truth table for* $(p \Rightarrow q) \Rightarrow (p \wedge q)$.

Solution. The truth table is

p	q	$p \Rightarrow q$	$p \wedge q$	$(p \Rightarrow q) \Rightarrow (p \wedge q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

□

Exercise 5. *Write down the truth table for* $(p \wedge q) \Rightarrow (p \vee q)$ *and draw a conclusion.*

Solution. The truth table is

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \Rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Hence $(p \wedge q) \Rightarrow (p \vee q)$ is a tautology. We can also check this fact from the laws of logic. Since $\alpha \Rightarrow \beta \equiv \neg\alpha \vee \beta$, then $(p\wedge q) \Rightarrow (p \vee q) \equiv \neg(p\wedge q) \vee (p \vee q) \equiv (\neg p \vee \neg q) \vee (p \vee q) \equiv (p \vee \neg p) \vee (q \vee \neg q)$, and $p \vee \neg p$ and $q \vee \neg q$ are both tautologies, so $(p \vee \neg p) \vee (q \vee \neg q)$ is. □

Exercise 6. *Write down the truth table for* $(p \Leftrightarrow \neg q) \Leftrightarrow (q \Rightarrow p)$.

Solution. The truth table is

p	q	$\neg q$	$p \Leftrightarrow \neg q$	$q \Rightarrow p$	$(p \Leftrightarrow \neg q) \Leftrightarrow (q \Rightarrow p)$
T	T	F	F	T	F
T	F	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F

□

Exercise 7. *Use a truth table to show that* $\neg(p \Rightarrow q)$ *is logically equivalent to* $p \wedge \neg q$.

Solution. The truth table is

p	q	$\neg q$	$p \Rightarrow q$	$\neg(p \Rightarrow q)$	$p \wedge \neg q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

Since fifth and sixth columns are identical, then $\neg(p \Rightarrow q)$ is logically equivalent to $p \wedge \neg q$. □

Exercise 8. *Use the laws of logic to show that* $(p \wedge q) \vee \neg p \equiv \neg p \vee q$.

Solution. Using the laws of logic we have:

$$\begin{aligned}
(p \wedge q) \vee \neg p &\equiv \neg p \vee (p \wedge q) && \text{(Commutative law)} \\
&\equiv (\neg p \vee p) \wedge (\neg p \vee q) && \text{(Distributive law)} \\
&\equiv \neg p \vee q && \text{(Using that } (\neg p \vee p) \text{ is a tautology)}
\end{aligned}$$

□

Exercise 9. *Use the laws of logic to simplify* $\big(s \vee (p \wedge r \wedge s)\big) \wedge \big(p \vee (p \wedge q \wedge \neg r) \vee (p \wedge q)\big)$.

Solution. Notice that, by Absorption Law, $s \vee (p \wedge r \wedge s) \equiv s$. Also by Absorption Law, $p \vee (p \wedge q \wedge \neg r) \vee (p \wedge q) \equiv p$. Hence

$$\big(s \vee (p \wedge r \wedge s)\big) \wedge \big(p \vee (p \wedge q \wedge \neg r) \vee (p \wedge q)\big) \equiv s \wedge p.$$

□

Exercise 10. *Definition: A compound statement is* **satisfiable** *if there is an assignment of truth values that makes it true. When no such assignment exists, i.e., when the compound statement is false for all assignments of truth values, then the compound proposition is* **unsatisfiable**. *In this sense, one can say that finding a particular assignment of truth values that makes a compound statement true amounts to giving a solution to this compound statement.*

a) Show that a compound proposition is unsatisfiable if and only if its negation is true for all assignments of truth values to the variables, i.e., if and only if its negation is a tautology.

b) Use truth tables to determine which of the following compound propositions is satisfiable, a tautology, or unsatisfiable: (i) $p \Rightarrow (p \vee q)$, *(ii)* $\neg(p \Rightarrow (p \vee t))$

Solution. a) A compound proposition is unsatisfiable if and only if there is no assignment of truth values such that the proposition is true. This is equivalent to say that for all assignments of truth values, the proposition is false. Since the negation of a proposition corresponds to take the opposite truth values, we have that a compound proposition is unsatisfiable if and only if for all assignments of truth values, the negation of the proposition is true. This is what we wanted to show.

We construct the truth tables as follows:

p	q	$p \vee q$	$p \Rightarrow (p \vee q)$	$\neg(p \Rightarrow (p \vee q))$
T	T	T	T	F
T	F	T	T	F
F	T	T	T	F
F	F	F	T	F

We conclude then that $p \Rightarrow (p \vee q)$ is a tautology, and $\neg(p \Rightarrow (p \vee q))$ is unsatisfiable. □