MA0301
ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2020

Set 10

Exercise 1. *Lewis–Zax (page 37): exercise 3.6. For any $n \geq 0$, let*

$$S(n) = \sum_{i=0}^{n} 2^{-i}.$$

We want to show that $S(n)$ is always less than 2, but becomes as close as we wish to 2 if n is sufficiently large.

(1) *What are $S(0), S(1), S(2)$, and $S(3)$?*
(2) *Conjecture a general formula for $S(n)$ of the form $S(n) = 2 - \ldots$.*
(3) *Prove by induction that the formula is correct for all $n \geq 0$.*
(4) *Now let ϵ be a small positive real number. How big does n have to be for $S(n)$ to be between $2 - \epsilon$ and 2?*

Exercise 2. *Lewis–Zax (page 67): exercise 6.2.*

Exercise 3. *Lewis–Zax (page 108): exercises 10.1., 10.3. (Here the exclusive OR is defined to be $p \oplus q := (p \vee q) \wedge (\neg p \vee \neg q)$. Note: In 10.1 (i), consider instead $\neg p \vee (q \wedge \neg p)$.*

Exercise 4. *Consider the set $X := \{a_1, a_2, a_3, a_4\}$. Draw the Hasse diagram corresponding to the partial order*

$$R := \{(a_1, a_1), (a_2, a_2), (a_3, a_3), (a_4, a_4), (a_1, a_2), (a_2, a_3), (a_1, a_3), (a_3, a_4), (a_1, a_4), (a_2, a_4)\}$$

defined on X.

Exercise 5. *For non-negative integers n, k, we define the numbers:*

- $N(0, 0) = 1,$
- $N(n, k) = 0, \quad$ *if* $k > n > 0,$ *or* $n > 0, k = 0,$
- $N(n, k) = \dfrac{1}{n}\dbinom{n}{k}\dbinom{n}{k-1}, \quad n \geq k \geq 1.$

Show that $N(n, n+1-k) = N(n, k)$, for $n > 0$ and $k \in \{0, 1, 2, \ldots, n\}$. These are the Narayana numbers.

Exercise 6. *Let $p_1, p_2, \ldots, p_{12}$ be twelve given points in the plane. Suppose that no three of the points are on the same line. How many triangles, having vertices among the points $p_1, p_2, \ldots, p_{12}$ and one of which is p_1 are there? Provide a detailed computations.*

Exercise 7. *Is there an undirected graph with 102 vertices, such that exactly 49 vertices have degree 5, and the remaining 53 vertices have degree 6?*

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Exercise 8. *If a connected planar graph has 12 vertices, each of degree 3, how many regions and edges does the graph have?*

Exercise 9. *Let $G = (V, E)$ be an undirected connected loop-free graph. Suppose that G is planar and determines $r = 53$ regions. If for some planar embedding of G, each region has at least five edges in its boundary, show that $|V| > 81$.*

Exercise 10. *If $G = (V, E)$ is connected graph with $|E| = 17$ and $\deg(v) > 2$ for all $v \in V$, what is the maximum value of $|V|$?*