MA0301
ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2019

Exercise Set 8

NOTE: Problems marked with a $\star$ are mandatory. Their solutions must be included to get the set approved.

$\star$ **Exercise 1.** *Define the relations $R, S \subset \mathbb{N} \times \mathbb{N}$:*

$$R = \{(0,2),(0,5),(0,9),(1,9),(1,12),(1,15),(2,2)\}$$

$$S = \{(2,0),(2,6),(5,6),(9,8),(12,1),(12,7),(15,4)\}.$$

1)Determine R^{-1}, S^{-1} and $(S \circ R)^{-1}$.

2) Can you deduce any connection between the three relations?

Solution. $R^{-1} = \{(2,0),(5,0),(9,0),(9,1),(12,1),(15,1),(2,2)\}$

$S^{-1} = \{(0,2),(6,2),(6,5),(8,9),(1,12),(7,12),(4,15)\}$

For $(S \circ R)^{-1}$, we first compute $S \circ R$, which equals

$$\{(2,2),(2,5),(2,9),(12,9),(12,12),(12,15)\}$$

so that

$$(S \circ R)^{-1} = \{(2,2),(5,2),(9,2),(9,12),(12,12),(15,12)\}.$$

We see that $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$. ■

$\star$ **Exercise 2.** *We define the three functions:*

$$f\colon \mathbb{Q} \to \mathbb{Q}, \quad x \mapsto f(x) = \frac{x}{3} \tag{1}$$

$$g\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto g(x) = x^3 \tag{2}$$

$$h\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (x,y) \mapsto h(x,y) = xy. \tag{3}$$

1) Describe the function $g \circ f$ and determine its domain and range.

2) Describe the function $g \circ h$ and determine its domain and range.

3) What can you say about $g \circ f = f \circ g$?

4) Can you form the compositions $h \circ f$ and $f \circ g$?

Solution. For **1)**, the domain is $\mathbb{Q}$ and the range is all fractions in $\mathbb{Q}$ for which the numerator and denominator can be written as cubes. The composition makes sense as $\mathbb{Q} \subseteq \mathbb{R}$.

For **2)**, the domain is $\mathbb{R} \times \mathbb{R}$ and the range is $\mathbb{R}$, as every real number has a real cube root (easy to see this if we set on of x and y equal to 1).

For **3)**, we note that the claimed equality doesn't make sense. We can only compose function if the ranges and domains of the functions are compatible. We can alter the involved functions

Date: March 16, 2019.

so that this is more sensible. Indeed, if we restrict g to $\mathbb{Q}$, the composition makes sense in either order, but then the claim is false as $\frac{x^3}{3} \neq \frac{x^3}{3^3}$.

For **4)**, we note that neither composition makes sense/can't be formed: the latter as $\mathbb{R} \not\subseteq \mathbb{Q}$, and the former as $\mathbb{Q} \not\subseteq \mathbb{R} \times \mathbb{R}$. ■

⋆ **Exercise 3.** *We define the three functions:*

$$a\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto a(x) = x + 2 \tag{4}$$

$$b\colon \mathbb{R}_+ \to \mathbb{R}, \quad x \mapsto b(x) = \sqrt{x} = x^{1/2} \tag{5}$$

$$c\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (x, y) \mapsto c(x, y) = \frac{x}{y}. \tag{6}$$

1) Determine the ranges of a, b, c.
2) Which of the three functions are injective?
3) Which of the three functions are surjective?
4) Which of the three functions are bijective?

Solution. **1)**: a has range equal to $\mathbb{R}$ as for given y we can choose $x = y - 2$. b has range equal to $\mathbb{R}_+$ as every positive real number has a real square root. Finally c has range equal to $\mathbb{R}$ as we can choose y equal to 1.

2): both a and b are injective, while c is not as $c(2x, 2y) = c(x, y)$.

3): a and c are surjective, while b is not. No negative numbers are in the range of b.

4): by the preceding two parts, only a is both injective and surjective and hence bijective. ■

⋆ **Exercise 4.** *Define $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$. Consider the following relation C on $\mathbb{R}^2\times\mathbb{R}^2$: $\big((a_1, b_1), (a_2, b_2)\big) \in C$ iff $a_1^2 + b_1^2 = a_2^2 + b_2^2$. Does C define an equivalence relation on $\mathbb{R}^2 \times \mathbb{R}^2$? If yes, determine the equivalence classes of R.*

Solution. We solve this using Exercise 10 and the fact that we can define a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ by setting $f(a, b) = a^2 + b^2$. We see that the equivalence relation defined by f on $\mathbb{R}^2 \times \mathbb{R}^2$ is the same as the relation C. ■

⋆ **Exercise 5.** (Grimaldi, 5. ed., Exercises 5.7, page 288) *Exercise 17*

Solution. **a)**: The range of f is $\{2, 3, 4, \ldots\}$.

b): From **a)** we see that f is not onto.

c): It is one to one. Indeed, if $f(x) = f(y)$ then $x + 1 = y + 1$ so that $x = y$.

d): The range of g is all of $\mathbb{Z}^+$.

e): From **d)** we see that g is onto.

f): Since $g(1) = 1 = g(2)$, we see that g is not one-to-one.

g): We note that $g(x) = x - 1$ on the range of f. Hence, $(g \circ f)(x) = g(f(x)) = g(x + 1) = x + 1 - 1 = x$.

h: $(f \circ g)(x) = 2, 3, 4, 7, 12, 25$ for $x = 2, 3, 4, 5, 7, 12$ respectively, as for each of these x, we see that $g(x) = x - 1$.

i: **b)** shows that f is not onto, yet **g)** and **h)** seem to suggest that it has an inverse, a contradiction to Theorem 5.8. However, f only has a *left* inverse: $f \circ g$ is not surjective, as 1 is not in its range. ■

⋆ **Exercise 6.** *We consider the universe U and the set $S \subset U$. The characteristic function $f_S : U \to \{0,1\}$ of the set S is defined by*

$$f_S(x) := \begin{cases} 1, & x \in S \\ 0, & \text{else} \end{cases}$$

Let A and B be sets in U. Show that $\forall x \in U$

$$f_{A\cup B}(x) = f_A(x) + f_B(x) - (f_A f_B)(x).$$

Solution. One can do this by looking at different cases: if $x \in A$ and $x \in B$, then

$$f_{A\cup B}(x) = 1 = 1 + 1 - 1 \cdot 1 = f_A(x) + f_B(x) - (f_A f_B)(x).$$

If $x \in A$ but $x \notin B$, then

$$f_{A\cup B}(x) = 1 = 1 + 0 - 1 \cdot 0 = f_A(x) + f_B(x) - (f_A f_B)(x).$$

Since the problem is symmetric in A and B, this also shows that the claim holds for $x \notin A$ and $x \in B$.

Finally, if $x \notin A$ and $x \notin B$, then

$$f_{A\cup B}(x) = 0 = 0 + 0 - 0 \cdot 0 = f_A(x) + f_B(x) - (f_A f_B)(x).$$

■

⋆ **Exercise 7.** *Show that for $n \in \mathbb{N}$ we have the relation between the Fibonacci and Lucas numbers:*

$$5F_{n+2} = L_{n+4} - L_n.$$

Note that you can find the definition of the Fibonacci numbers and the Lucas numbers in earlier exercise sets or in the book.

Solution. Recall that $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$; and also that $L_0 = 2$, $L_1 = 1$ and that $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, so that $L_2 = 3$, $L_3 = 4$ and $L_4 = 7$.

We proceed by induction. The base step is $5 \cdot F_2 = 5 = L_4 - L_0 = 7 - 2$ and $5 \cdot F_3 = 10 = L_5 - L_1 = 11 - 1$, which is true.

We assume that it holds for the cases $k \leq n$ and aim to show the claim for the $n+1$-case, i.e. we aim to use strong induction. A priori, we don't know that we will find this useful, but it is likely given the form of the defining recurrence relations for the Fibonacci and Lucas numbers.

$$\begin{aligned} 5F_{n+1+2} &= 5F_{n+2} + 5F_{n+1} \\ &= L_{n+4} - L_n + L_{n-1+4} - L_{n-1} \\ &= L_{n+4} + L_{n-1+4} - L_n - L_{n-1} \\ &= L_{n+1+4} - L_{n+1} \end{aligned}$$

Here we used the induction hypothesis in the second equality. ■

⋆ **Exercise 8.** *A positive integer m is divisible by 3 if $m = 3t$ for a positive integer t.*

Show by induction that the integer $n^3 - n$ is divisible by 3 for any positive integer n.

Solution. The base step holds for $n = 1$ as $1^3 - 1 = 0$ is divisible by 3.

For the induction step, we assume the claim holds for n, and aim to show that it holds for $n+1$.

$$\begin{aligned}(n+1)^3 - n - 1 &= n^3 + 3n^2 + 3n + 1 - n - 1 \\ &= n^3 - n + 3n^2 + 3n \\ &= (n^3 - n) + 3n^2 + 3n\end{aligned}$$

By the induction hypothesis, $(n^3 - n)$ is divisible by 3, and so the result follows. ■

Exercise 9. (Grimaldi, 5. ed., Exercises 15.4, page 741) *Exercise 6*

Solution. This is a duplicate from last week's Exercise set! ■

Exercise 10. *Let A is a non-empty set, which is the domain of the function f. Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$. Is R an equivalence relation on A? If yes, determine the equivalence classes of R.*

Solution. We try and check each of the properties of an equivalence relation for R.

Reflexivity holds since $f(x) = f(x)$ holds for every $x \in A$ as equality is reflexive.

Symmetry holds as if $f(x) = f(y)$ then $f(y) = f(x)$ as equality is symmetric.

Transitivity holds as if $f(x) = f(y)$ and $f(y) = f(z)$, then $f(x) = f(z)$ by transitivity of equality. ■

Exercise 11. *Use the rules of inference to show that the following statement is true:*

$$((p \wedge \neg q) \to s) \wedge \neg(\neg p \vee q) \wedge (\neg z \to \neg s) \wedge (\neg z \vee r) \to r$$

Solution. We assume $(p \wedge \neg q) \to s), \neg(\neg p \vee q), (\neg z \to \neg s)$ and $(\neg z \vee r)$ and try to deduce r using rules of inference and logical equivalences.

$\neg(\neg p \vee q)$ is equivalent to $p \wedge \neg q$ by DeMorgan's and double negation.

From this and $(p \wedge \neg q) \to s$ we can deduce s using Modus Ponens, which together with the contrapositive of $\neg z \to \neg s$ and Modus Ponens implies z. As $\neg z \vee r$ is equivalent to $z \to r$, we deduce r using Modus Ponens again. ■

Exercise 12. *Let A, B be arbitrary sets. Use the laws of set theory to show that:*

$$\text{If} \quad (A \cup B) \subseteq (A \cap B) \quad \text{then} \quad A = B.$$

Solution. $(A \cup B) \subseteq (A \cap B)$ implies that $((A \cap A) \cup (A \cap B)) \subseteq (A \cap A \cap B)$ by distributivity of $\cap$ over $\cup$.

Now we compute

$$((A \cap A) \cup (A \cap B)) = A \cup (A \cap B) = A \subseteq (A \cap A \cap B) = A \cap B \subseteq A$$

Hence, $A = A \cap B$. As the problem is symmetric in A and B, we also deduce $B = A \cap B$ and so $A = B$. ■