MA0301
ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2019

Exercise Set 6

NOTE: Problems marked with a $\star$ are mandatory. Their solutions must be included to get the set approved.

$\star$ **Exercise 1.** *Recall that binary relations are sets, such that the set operations $\cup$, $\cap$ and complement apply to them. Let A, B be two non-empty sets. Let $R \subseteq A \times B$ be a binary relation. We denote the domain of R by $\mathrm{dom}(R)$ and the range of R by $\mathrm{ran}(R)$. The complement of R is defined as $\bar{R} := (A \times B) \backslash R = (A \times B) - R$.*

Now let $R_1, R_2 \subseteq A \times B$ be two binary relations. Show that:
i) $\mathrm{dom}(R_1 \cup R_2) = \mathrm{dom}(R_1) \cup \mathrm{dom}(R_2)$
ii) $\mathrm{ran}(R_1 \cup R_2) = \mathrm{ran}(R_1) \cup \mathrm{ran}(R_2)$
iii) $\mathrm{dom}(R_1 \cap R_2) \subseteq \mathrm{dom}(R_1) \cap \mathrm{dom}(R_2)$
iv) $\mathrm{ran}(R_1 \cap R_2) \subseteq \mathrm{ran}(R_1) \cap \mathrm{ran}(R_2)$.

Solution. We show i) as the proof of ii) is done by simply replacing "domain" everywhere by "range": note that $(a, b) \in R_1 \cup R_2 \Leftrightarrow ((a, b) \in R_1 \vee (a, b) \in R_2)$. Now, $a \in \mathrm{dom}(R_1 \cup R_2)$ iff there is some (a, b) such that $(a, b) \in R_1 \cup R_2$ iff $(a, b) \in R_1 \vee (a, b) \in R_2$ iff $(a \in \mathrm{dom}(R_1) \vee a \in \mathrm{dom}(R_2))$. All together, this gives the first equivalence in the following:

$$a \in \mathrm{dom}(R_1 \cup R_2) \Leftrightarrow (a \in \mathrm{dom}(R_1) \vee a \in \mathrm{dom}(R_2)) \Leftrightarrow (a \in \mathrm{dom}(R_1) \cup \mathrm{dom}(R_2)).$$

The second follows by definition of $\cup$.

As for iii) and iv), we only show iii) as the proof of iv) follows as above: $a \in \mathrm{dom}(R_1 \cap R_2)$ iff there is some (a, b) such that $(a, b) \in R_1 \cap R_2$ iff $(a, b) \in R_1 \wedge (a, b) \in R_2$. The last claim implies that $a \in \mathrm{dom}(R_1) \wedge a \in \mathrm{dom}(R_2)$, which is equivalent to $a \in \mathrm{dom}(R_1) \cap \mathrm{dom}(R_2)$. ■

$\star$ **Exercise 2.** *You have the set $A = \{a, b, c, d, e, f\}$. We define the following two binary relations on A:*

$$S_1 = \{(a, b), (b, c), (c, d), (d, e), (e, f), (f, a)\}$$
$$S_2 = \{(a, b), (b, c), (c, d), (d, e), (e, f), (f, a), (a, d), (d, a), (f, c), (c, f)\}.$$

1) Draw the arrow diagrams representing these two binary relation.
2) Give the arrow diagram representations for:
(a) $S_1 \cap S_2$, *(b)* $S_1 \cup S_2$, *(c)* $S_1 - S_2$, *(d)* $\bar{S}_1$ *(e)* S_1^{-1} *(f)* $S_1 \circ S_2$.

Solution. Solution omitted. ■

$\star$ **Exercise 3.** *Consider the set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Define the relation R on S that relates every number in S to those that have the same number of divisors as it. Show that R is an equivalence relation. Find the partition of S corresponding to R.*

Date: April 24, 2019.

Solution. While this problem is fairly concrete, it is perhaps easier to show that this is an equivalence relation by working "abstractly": every number x has the same number of divisor as itself, i.e. $(x, x) \in R$ for every $x \in S$, and hence reflexivity follows; (x, y) is in the relation iff x and y have the same number of divisors iff (y, x) is in the relation, and hence we get symmetry; and finally, for transitivity, note that $(x, y) \in R$ and $(y, z) \in R$ iff x has the same number of divisors as y and y has the same number of divisors as z, so that by the transitivty of equality for whole numbers, x has the same number of divisors as z, implying $(x, z) \in R$, and hence we are done.

We note here that we consistently used that equality for whole numbers is an equivalence relation to show that R is one, too. This is often useful in similar situations.

Finding the partition: 1 is the only number with only 1 divisor.

$2, 3, 5, 7, 11$ are the only primes and thus have 2 divisors. While $4, 8, 9$ are composite, they likewise only have 2 divisors, being squares and cubes.

$6, 10, 12$ are all composite with 3 divisors.

The partition is thus given as $\{1\}, \{2, 3, 4, 5, 7, 8, 9, 11\}$ and $\{6, 10, 12\}$. ■

$\star$ **Exercise 4.** *Let R_1 and R_2 be two equivalence relations on the set A. Show that $R_1 \circ R_2$ is an equivalence relation if and only if $R_1 \circ R_2 = R_2 \circ R_1$.*

Solution. $R_1 \circ R_2$ is reflexive iff $(x, x) \in R_1 \circ R_2$ for all x in A iff $(x, y) \in R_1$ and $(y, x) \in R_2$ for all x in A and for some $y \in A$. But then, as both R_1 and R_2 are equivalence relations, we can choose y equal to x for all x in A.

$R_1 \circ R_2$ is symmetric iff $(x, y) \in R_1 \circ R_2 \Leftrightarrow (y, x) \in R_1 \circ R_2$. Now $(x, y) \in R_1 \circ R_2$ iff there is some $(x, z) \in R_1$ and $(z, y) \in R_2$. Using the symmetric property for R_1 and R_2, we get that this is equivalent to $(z, x) \in R_1$ and $(y, z) \in R_2$, which is equivalent to $(y, x) \in R_2 \circ R_1$. Repeating this argument with x and y interchanged we get that $(y, x) \in R_1 \circ R_2$ iff $(x, y) \in R_2 \circ R_1$. Hence, $R_1 \circ R_2$ is symmetric iff $R_1 \circ R_2 = R_2 \circ R_1$.

$R_1 \circ R_2$ is transitive iff $(x, y) \in R_1 \circ R_2$ and $(y, z) \in R_1 \circ R_2$ implies $(x, z) \in R_1 \circ R_2$.

Note that: $(x, y) \in R_1 \circ R_2$ is equivalent to there being a x' such that $(x, x') \in R_1$ and $(x', y) \in R_2$. $(y, z) \in R_1 \circ R_2$ is equivalent to there being a y' such that $(y, y') \in R_1$ and $(y', z) \in R_2$.

If $R_1 \circ R_2 = R_2 \circ R_1$, then $(x, y) \in R_2 \circ R_1 = R_1 \circ R_2$ iff $(x, x'') \in R_2$ and $(x'', y) \in R_1$ and $(y, z) \in R_1 \circ R_2$ iff $(y, y'') \in R_2$ and $(y'', z) \in R_1$.

$(x', y') \in R_2 \circ R_1 = R_1 \circ R_2$ follows from the above, and is equivalent to $(x', w) \in R_1$ and $(w, y') \in R_2$, from which we deduce $(x, w) \in R_1$ and $(w, z) \in R_2$, which is equivalent to $(x, z) \in R_1 \circ R_2$.

This was what was to be shown. (It might seem that we have skipped the "only if" direction when dealing with the transitivity property, but it is not necessary: the necessity of the assumption was shown when dealing with the symmetric property.)

Digression: Upon reflection, we might see that what we have shown is that $(R_1 \circ R_2) \circ (R_1 \circ R_2) \subseteq R_1 \circ R_2$. Hence, as R_1 and R_2 are equivalence relations, what we have done, essentially, is to "show" that

$$\begin{aligned}
(R_1 \circ R_2) \circ (R_1 \circ R_2) &= R_1 \circ (R_2 \circ R_1) \circ R_2 \\
&= R_1 \circ (R_1 \circ R_2) \circ R_2 \\
&= (R_1 \circ R_1) \circ (R_2 \circ R_2) \\
&\subseteq R_1 \circ R_2
\end{aligned}$$

Note, that the equations/computations above don't necessarily make formal sense, but are meant to express the intuition for this. ■

Exercise 5. (Grimaldi, 5. ed., Exercises 5.1, page 252) *Exercise 9*

Solution. This is in S-28 in the solutions at the end of the book. ■

Exercise 6. (Grimaldi, 5. ed., Exercises 5.1, page 252) *Exercise 11*

Solution. This is in S-28 in the solutions at the end of the book. ■

Exercise 7. (Grimaldi, 5. ed., Exercises 7.1, page 343) *Exercise 5*

Solution. This is in S-40 in the solutions at the end of the book. ■

⋆ **Exercise 8.** *Which of the relations in Exercise 5* (Grimaldi, 5. ed., Exercises 7.1, page 343) *are equivalence relations?*

Solution. Only the relations in **c)** and **f)** are equivalence relations as ■

⋆ **Exercise 9.** *Show in detail the set equality*

$$(A \triangle B) \cup B = (A \cup B).$$

Solution.

$$\begin{aligned}
(A \triangle B) \cup B &= ((A \cap \bar{B}) \cup (B \cap \bar{A})) \cup B \\
&= ((A \cap \bar{B}) \cup B) \cup ((B \cap \bar{A}) \cup B) \\
&= ((A \cup B) \cap (\bar{B} \cup B)) \cup ((B \cup B) \cap (\bar{A} \cup B)) \\
&= (((A \cup B) \cap (\mathcal{U}))) \cup (B \cap (\bar{A} \cup B)) \\
&= (A \cup B) \cup ((B \cap (\bar{A} \cup B)))) \\
&= A \cup B
\end{aligned}$$

■

⋆ **Exercise 10.** *Use the laws of logic to simplify* $\big(p \vee (p \wedge q) \vee (p \wedge q \wedge \neg r)\big) \wedge \big((p \wedge r \wedge t) \vee t\big)$.

Solution.

$$\big(p \vee (p \wedge q) \vee (p \wedge q \wedge \neg r)\big) \wedge \big((p \wedge r \wedge t) \vee t\big) \Leftrightarrow p \wedge t$$

Here, we have used three times that $p \vee (p \wedge q) \Leftrightarrow p$ for propositional variables p and q. ■

⋆ **Exercise 11.** *Show by induction that*

$$\sum_{k=1}^{n} 4(k^3 - 3k^2 + 2k) = (n^2 + n)(n^2 - 3n + 2).$$

Solution. For the base step, we note that both sides equal zero for $n = 1$.

For the induction step, assume the claim holds for n. We want to show it then holds for $n + 1$, too.

$$
\begin{aligned}
LHS &= 4((n+1)^3 - 3(n+1)^2 + 2(n+1)) + \sum_{k=1}^{n} 4(k^3 - 3k^2 + 2k) \\
&= \sum_{k=1}^{n+1} 4(k^3 - 3k^2 + 2k)
\end{aligned}
$$

$$
\begin{aligned}
RHS &= 4((n+1)^3 - 3(n+1)^2 + 2(n+1)) + (n^2+n)(n^2-3n+2) \\
&= 4(n+1)^3 - 12(n+1)^2 + 8(n+1) + n^4 - 3n^3 + 2n^2 + n^3 - 3n^2 + 2n \\
&= ((n+1)^2 + (n+1))((n+1)^2 - 3(n+1) + 2)
\end{aligned}
$$

■

⋆ **Exercise 12.** *Show that if u_n is defined recursively by the rules $u_1 = 1$, $u_2 = 5$ and for all $n > 1$, $u_{n+1} = 5u_n - 6u_{n-1}$, then $u_n = 3^n - 2^n$ for all $n \in \mathbb{N}$.*

Solution. We proceed by induction.

The base step is clear: $u_3 = 5u_2 - 6u_1 = 25 - 6 = 19 = 3^3 - 2^3$.

For the inductive step, we compute:

$$
\begin{aligned}
u_{n+1} = 5u_n - 6u_{n-1} &= 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1}) \\
&= 5 \cdot 3^n - 5 \cdot 2^n - 2 \cdot 3^n + 3 \cdot 2^n \\
&= 3^{n+1} - 2^{n+1}.
\end{aligned}
$$

This was what was to be shown. ■

⋆ **Exercise 13.** (Grimaldi, 5. ed., Exercises 4.2, page 209) *Exercise 12*

Solution. For **a)**, we need some known results on sums of angles for trigonometric functions:

$$\cos(\sigma + \theta) = \cos\sigma\cos\theta - \sin\theta\sin\sigma$$

and

$$\sin(\sigma + \theta) = \sin\sigma\cos\theta + \cos\sigma\sin\theta.$$

We then compute

$$
\begin{aligned}
(\cos\theta + i\sin\theta)^2 &= (\cos^2\theta - \sin^2\theta) + 2i(\cos\theta\sin\theta) \\
&= \cos 2\theta + i\sin 2\theta,
\end{aligned}
$$

where for the left summand we used the first identity, and for the right the second.

For **b)**, we note that $n = 1, 2$ hold either trivially or by **a)**.

For the induction step, we assume the claim holds in the n case and use the identities above.

$$
\begin{aligned}
(\cos\theta + i\sin\theta)^{n+1} &= (\cos n\theta + i\sin n\theta)(\cos\theta + i\sin\theta) \\
&= (\cos n\theta\cos\theta - \sin n\theta\sin\theta) + i(\cos n\theta\sin\theta + \sin n\theta\cos\theta) \\
&= \cos((n+1)\theta) + i\sin((n+1)\theta)
\end{aligned}
$$

Hence, the induction goes through, and the claim follows. ■

Exercise 14. (Grimaldi, 5. ed., Exercises 4.2, page 209) *Exercise 13*

Solution. This is in S-18 in the solutions at the end of the book. ■