MA0301 ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2019

Exam 1

Exercise 1 (Sets):	15 points
Exercise 2 (Relations):	15 points
Exercise 3 (Induction):	20 points
Exercise 4 (Functions):	15 points
Exercise 5 (Graphs):	15 points
Exercise 6 (Boolean algebra):	10 points
Exercise 7 (Finite state automata)	10 points

Total: 100 points

Note: In each of the exercises 1.1, 2.1, 4.1, 5.1, exactly one answer is correct.

Exercise 1. Sets (15 points)

(1) **(3 points)** *Suppose* $U = \mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$. *Let* $A = \{1, 2, 3, 4\}$ *and* $B := \{2, 3, 8, 9\}$.

Which of the three statements is correct?

I) The symmetric difference $A \triangle B = \{1, 4, 8, 9\}$.

II) The symmetric difference $A \triangle B = U - (A \cup B)$.

III) The symmetric difference $A \triangle B = \{2, 3\}$.

(2) **(5 points)** *A* fundamental product *of the sets* $A_1, A_2, \ldots, A_n$ *is defined to be a set of the form* $A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap \cdots \cap A_n^{\epsilon_n}$, *where* $A_i^{\epsilon_i}$ *is either the set* A_i *or its complement* $\overline{A_i}$.

a) *List all fundamental products of three sets* A_1, A_2, A_3

b) *Find the number of fundamental products of* m *sets* $A_1, A_2, \ldots, A_m$

(3) **(7 points)** *Write down the definition of the cartesian product of two sets. Show that for three sets* A, B, C *the following equality of sets holds:*

$$A \times (B - C) = (A \times B) - (A \times C).$$

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Exercise 2. Relations (15 points)

(1) **(3 points)** *Which of the three statements is correct?*

I) A partial order P on a set X is reflexive, symmetric, and transitive.

II) A partial order P on a set X is reflexive, anti-symmetric, and transitive.

III) A partial order P on a set X is anti-reflexive, anti-symmetric, and transitive.

(2) **(4 points)** *Show that:*

$$R = \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(a,c),(a,d),(a,e),(b,c),(e,d),(e,c)\}$$

defines a partial order on $A := \{a,b,c,d,e\}$ and draw the corresponding Hasse diagram.

(3) **(4 points)** *Let $B := \{2,3,4,16\}$ be ordered by divisibility. Find the maximal and minimal elements of B.*

(4) **(4 points)** *Let $C := \{a,b,c,d,e,f,g,h\}$. For each of the following families of subsets of C, determine whether or not it is a partition of C. If it is a partition of C draw the graphical representation of the corresponding equivalence relation.*

a) $\{\{a,c,e,g\},\{b,d\},\{h,f,c\}\}$

b) $\{\{a,c,e,g\},\{d,f,c\}\}$

c) $\{\{a,c,e,g\},\{b,h\},\{d,f\}\}$

Exercise 3. Induction (20 points)

(1) **(5 points)** *Use induction to show that*

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}.$$

(2) **(7 points)** *Determine for which natural numbers we have* $n^2 \geq 2n+1$.

(3) **(8 points)** *Let* c_n *be the sequence defined by* $c_0 = 1 = c_1$, $c_2 = 3$ *and for* $n > 0$

$$c_{n+2} = 3c_{n+1} - 3c_n + c_{n-1}.$$

Show that for $n \geq 0$

$$c_n = n^2 - n + 1.$$

Exercise 4. Functions (15 points)

(1) **(3 points)** *Let A, B be sets. Suppose that f is a subset of $A \times B$. Which of the three statements is correct?*

I) The set f defines a function if and only if each element $a \in A$ appears as the first coordinate in at least one ordered pair of f.

II) The set f defines a function if and only if each element $b \in B$ appears as the second coordinate in exactly one ordered pair of f.

III) The set f defines a function if and only if each element $a \in A$ appears as the first coordinate in exactly one ordered pair of f.

(2) **(5 points)** *Define the function $f(x) := 2x - 3$ from $\mathbb{R}$ to $\mathbb{R}$. Show that F is surjective and injective. Find its inverse function f^{-1}.*

(3) **(7 points)** *Let $f : A \to B$ and $g : B \to C$ be given. Show that if $g \circ f : A \to C$ is injective, then f is injective.*

Exercise 5. Graphs (15 points)

(1) **(3 points)** *Which of the three statements is correct?*

I) Let $G = G(V, E)$ be a (multi-)graph. Then G has a closed Euler trail if and only if it is connected and every vertex of G has either degree 3 or degree 5.

II) Let $G = G(V, E)$ be a (multi-)graph. Then G has a closed Euler trail if and only if it is connected and every vertex of G has even degree.

III) Let $G = G(V, E)$ be a (multi-)graph. Then G has a closed Euler trail if and only if it is connected and G has exactly two vertices of odd degree.

(2) **(5 points)** *Let $V = \{a, b, c, d\}$. Determine which of the following graphs $G = G(V, E)$ has an Euler trail or Euler circuit.*

a) $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\}$.
b) $E = \{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{d, a\}\}$.
c) $E = \{\{a, b\}, \{c, d\}, \{b, a\}, \{c, c\}, \{d, c\}\}$.

(3) **(7 points)** *Find the number of edges of the complete graph K_n, $n \geq 1$.*

Exercise 6. Boolean algebra (10 points)

(1) **(5 points)** *Let B be a Boolean algebra and let a be any element in B. Show that if $a + x = 1$ and $a \cdot x = 0$, then $x = \bar{a}$.*

(2) **(5 points)** *Let B be a Boolean algebra. Show that for any elements $a, b \in B$ we have that i) $a + b = b$ is equivalent to ii) $a \cdot b = a$.*

Exercise 7. Finite state automata (10 points)

(1) **(5 points)** *Let* $\Sigma := \{a, b\}$ *and define the language* $L := \{a^m b^n \mid m, n > 0\}$. *Construct an automaton* A' *which will accept the language* $L(A')$.

(2) **(5 points)**

a) *Draw the table for the automaton* A *in Fig. 1.*

b) *Find the language* $L(A)$ *accepted by the automaton* A *in Fig. 1.*

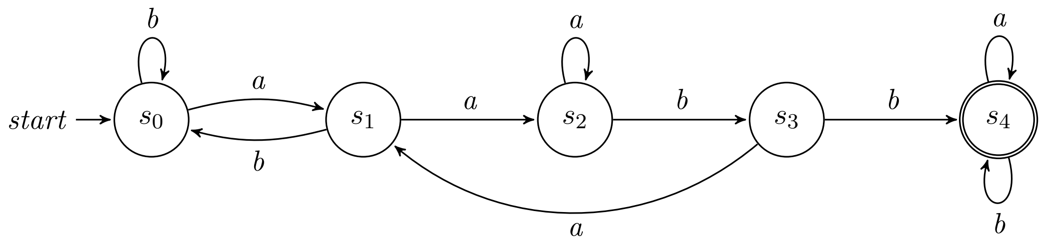

FIGURE 1. The automaton A.