

Wednesday 3 Feb 2016

Theorem: Let A, B sets in universe \mathcal{U} . Then the following are equivalent:

a) $A \subseteq B$

b) $A \cup B = B$

" A does not add anything new"

c) $A \cap B = A$

"everything in A is also in B "

d) $\overline{B} \subseteq \overline{A}$

Proof:

- (a) \Rightarrow (b): Let $x \in \mathcal{U}$ be arbitrary. If $x \in B$, then $x \in A \cup B$, hence $B \subseteq A \cup B$.

On the other hand, if $x \in A \cup B$ then $x \in A$ or $x \in B$.

- If $x \in B$,

- If $x \in A$, then $x \in B$ since $A \subseteq B$.

So we find that $x \in B$ in any case. Hence $A \cup B \subseteq B$.

This shows that $A \cup B = B$.

- (b) \Rightarrow (c): We already know that $A \cap B \subseteq A$. So let $x \in \mathcal{U}$ be arbitrary and suppose that $x \in A$. Then $x \in A \cup B$. As $A \cup B = B$, we find $x \in B$. Since $x \in A$ and $x \in B$, we have $x \in A \cap B$. Since x was arbitrary, this implies that $A \subseteq A \cap B$. Hence $A \cap B = A$.

- (c) \Rightarrow (d): Let $x \in \mathcal{U}$ be arbitrary and suppose that $x \notin \overline{B}$. Then $x \notin B$. Therefore, $x \notin A \cap B$. As $A \cap B = A$, we find $x \notin A$. So $x \in \overline{A}$. Since x was arbitrary, this means $\overline{B} \subseteq \overline{A}$.

- (d) \Rightarrow (a): Let $x \in \mathcal{U}$ be arbitrary and suppose $x \in A$. Then $x \notin \overline{A}$. Since $\overline{B} \subseteq \overline{A}$ and $x \notin \overline{A}$, we must have $x \notin \overline{B}$ (modus tollens). Since $x \notin \overline{B}$, we find $x \in B$. Since x was arbitrary, we find $A \subseteq B$.

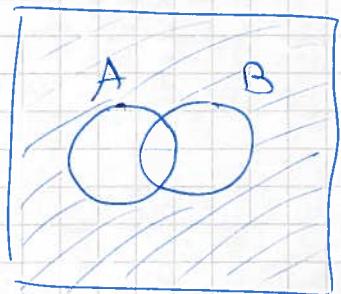
The laws of set theory: $A, B, C \subseteq \mathcal{U}$

1) $\overline{\overline{A}} = A$	Law of Double Complement DeMorgan's laws
2) $\overline{A \cup B} = \overline{A} \cap \overline{B}$	
3) $\overline{A \cap B} = \overline{A} \cup \overline{B}$	Commutative laws
4) $A \cup (B \cup C) = (A \cup B) \cup C$	
5) $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
6) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	
7) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
8) $A \cup A = A$	
9) $A \cap A = A$	Idempotent laws
10) $A \cup \emptyset = A$	
11) $A \cap \mathcal{U} = A$	Identity laws
12) $A \cup \overline{A} = \mathcal{U}$	
13) $A \cap \overline{A} = \emptyset$	Inverse laws
14) $A \cup \mathcal{U} = \mathcal{U}$	
15) $A \cap \emptyset = \emptyset$	Domination laws
16) $A \cup (A \cap B) = A$	
17) $A \cap (A \cup B) = A$	Absorption laws

Proof with \Leftrightarrow but give warning!

Proof: Let $x \in U$ be arbitrary.

$$\begin{aligned} \text{Then } x \in \overline{A \cup B} &\iff x \notin A \cup B \\ &\iff x \notin A \wedge x \notin B \\ &\iff x \in \overline{A} \wedge x \in \overline{B} \\ &\iff x \in \overline{A \cap B} \end{aligned}$$



Venn diagram

$$\text{Hence } \overline{A \cup B} = \overline{A \cap B}, \text{ as } x \text{ was arbitrary.}$$

Remark: We skip the concept of duality.

Remark: Venn diagrams work very well to get an idea of the problem and solution, but they are not proofs!
Also, they are less useful with more than 3 sets.

Remark: We may also use an analogue of truth tables to work out a given problem. These tables are called membership tables:

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

This "proves" that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ by writing out all options for elements to be in or not in A, B and C . However, this does not give much insight and becomes very much work if you have 7 sets or so.

Example: Using the laws of set theory: prove that $\overline{A \Delta B} = \overline{\overline{A} \Delta \overline{B}}$

$$\begin{aligned} \overline{A \Delta B} &= \overline{(A \cup B) \setminus (A \cap B)} \quad \text{by definition of } \Delta \\ &= \overline{(A \cup B)} \cap \overline{(A \cap B)} \quad \text{by definition of } \setminus \\ &= \overline{(A \cup B)} \cup \overline{(A \cap B)} \quad \text{by DeMorgan's law} \\ &= \overline{(A \cup B)} \cup \overline{(A \cap B)} \quad \text{by law of Double Complement} \\ &= (A \cap B) \cup \overline{(A \cup B)} \quad \text{by commutative law of } \cup \\ &= (A \cap B) \cup (\overline{A} \cap \overline{B}) \quad \text{by DeMorgan's law} \\ &= [(A \cap B) \cup \overline{A}] \cap [(A \cap B) \cup \overline{B}] \quad \text{by distributive law of } \cup \text{ over } \cap \\ &= [(A \cap \overline{A}) \cup (B \cap \overline{A})] \cap [(A \cap \overline{B}) \cup (B \cap \overline{B})] \quad \text{by distr. law of } \cup \text{ over } \cap \end{aligned}$$

$$\begin{aligned}
 &= [U \cap (B \cup \bar{A})] \cap [(A \cup \bar{B}) \cap U] && \text{by inverse law} \\
 &= [B \cup \bar{A}] \cap [A \cup \bar{B}] && \text{by identity law} \\
 &= (\bar{A} \cup B) \cap (A \cup \bar{B}) && \text{by commutative law of } \cup \\
 &= (\bar{A} \cup B) \cap \overline{\bar{A} \cap B} && \text{by DeMorgan's law} \\
 &= (\bar{A} \cup B) \setminus (\bar{A} \cap B) && \text{by definition of } \setminus \\
 &= \bar{A} \Delta B && \text{by definition of } \Delta
 \end{aligned}$$

Useful trick: $A \setminus B = A \cap \bar{B}$

Definition: Unions and intersections of more than 2 sets.

I non-empty set of indices
 \mathcal{U} universe.

For each $i \in I$, $A_i \subseteq \mathcal{U}$ a set.

Define $\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for at least one } i \in I\}$
 $\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$

Remark: We can use quantifiers to restate definition 10:

$$x \in \bigcup_{i \in I} A_i \iff \exists i \in I, x \in A_i$$

$$x \in \bigcap_{i \in I} A_i \iff \forall i \in I, x \in A_i$$

Example: $I = \mathbb{Z}^+, A_i = [-i, i]$.

Then $\bigcup_{i \in I} A_i = \bigcup_{i=1}^{\infty} A_i = \mathbb{R}$

and $\bigcap_{i \in I} A_i = \bigcap_{i=1}^{\infty} A_i = [-1, 1]$

Example: $I = \{2, 3, 4\}, A_i = \{i, i+1, i+2\}$

Then $\bigcup_{i \in I} A_i = \bigcup_{i=2}^4 A_i = \{2, 3, 4, 5, 6\}$

and $\bigcap_{i \in I} A_i = \bigcap_{i=2}^4 A_i = \{4\}$

Watch out: if $i = \{2, 7, 28\}$ then $\bigcup_{i \in I} A_i \neq \bigcup_{i=2}^{28} A_i$, in general.

Theorem: Generalized DeMorgan's laws: Let I be an index set and for each i , let $A_i \subseteq \mathcal{U}$ be a set.

a) $\overline{\bigcup_{i \in I} A_i} = \bigcap_{i \in I} \overline{A_i}$

b) $\overline{\bigcap_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}$

Proof: Let $x \in \mathcal{U}$ be arbitrary. (We prove only part (a) here.)

$$\begin{aligned} \text{Then } x \in \overline{\bigcup_{i \in I} A_i} &\iff \neg(x \in \bigcup_{i \in I} A_i) \\ &\iff \neg(\exists i \in I, x \in A_i) \\ &\iff \forall i \in I, \neg(x \in A_i) \\ &\iff \forall i \in I, x \in \overline{A_i} \\ &\iff x \in \bigcap_{i \in I} \overline{A_i} \end{aligned}$$

Hence $\overline{\bigcup_{i \in I} A_i} = \bigcap_{i \in I} \overline{A_i}$ since x was arbitrary.

2) Some more counting:

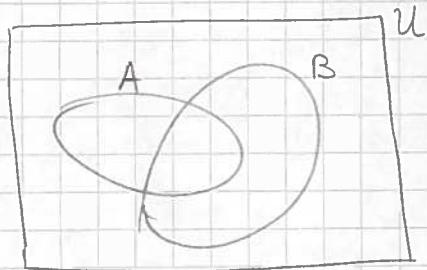
(We use Venn diagrams for intuition, but will prove our results neatly later on.)

Back to Billy the goat.

\mathcal{U} = all goats of Farmer McDonalds.

A = set of white goats

B = set of goats with horns.



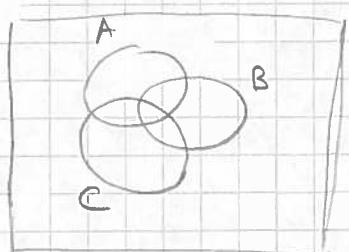
Then $A \cup B = \{ \text{goats that are white or have horns} \}$. Assume A and B are finite.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|\overline{A \cap B}| = |\overline{A \cup B}| = |\mathcal{U}| - |A \cup B| = |\mathcal{U}| - |A| - |B| + |A \cap B| \quad \text{for } \mathcal{U} \text{ finite}$$

Remark: For finite sets A and B in a universe \mathcal{U} :
 A and B are mutually disjoint if and only if $|A \cup B| = |A| + |B|$

Now let C = set of male goats. How do we count $|A \cup B \cup C|$?



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$\begin{aligned} |\overline{A \cap B \cap C}| &= |\overline{A \cup B \cup C}| = |\mathcal{U}| - |A \cup B \cup C| \\ &= |\mathcal{U}| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C| \end{aligned}$$

Notice: $|A| = |\overline{A \cap B \cap C}| + |\overline{A \cap B \cap C}| + |\overline{A \cap B \cap C}| + |\overline{A \cap B \cap C}|$

all these sets are disjoint from each other

Theorem:

The principle of inclusion and exclusion

Let S be a finite set, with $|S| = N$, and suppose we have conditions c_i for $1 \leq i \leq t$ which may be satisfied by some elements of S .

Write $N(c_i)$ for the number of elements of S that satisfies c_i , and $N(\bar{c}_1, c_2, \bar{c}_3)$ for the number of elements of S that satisfies c_2 , and not c_1 , and not c_3 . (Write $\bar{N} = N(\bar{c}_1, \bar{c}_2, \dots, \bar{c}_t)$ for the number of elements of S that satisfy none of the conditions c_i .)

$$\text{Then } \bar{N} = N - [N(c_1) + N(c_2) + \dots + N(c_t)]$$

$$+ [N(c_1, c_2) + N(c_1, c_3) + \dots + N(c_1, c_t) + N(c_2, c_3) + \dots + N(c_{t-1}, c_t)]$$

$$- [N(c_1, c_2, c_3) + \dots + N(c_{t-2}, c_{t-1}, c_t)]$$

:

$$+ (-1)^t N(c_1, c_2, \dots, c_t)$$

$$= N - \sum_{1 \leq i \leq t} N(c_i) + \sum_{1 \leq i < j \leq t} N(c_i, c_j) - \sum_{1 \leq i < j < k \leq t} N(c_i, c_j, c_k) + \dots + (-1)^t N(c_1, c_2, \dots, c_t)$$

Proof:

Let $x \in S$. We compare how many times x is counted on the left-hand side and the right-hand side of this equality.

- If x satisfies none of the conditions c_i , then x is counted in \bar{N} so x is counted 1 time on the left-hand side.
On the right-hand side, x is counted in N and nowhere else, so x is also counted once.
- If x satisfies k of the conditions c_i with $k \geq 1$, then x is not counted in \bar{N} . So x does not contribute to the left-hand side.

On the right-hand side:

- x is counted once in N
- and k times in $\sum_{1 \leq i \leq t} N(c_i)$
- and $\binom{k}{2}$ times in $\sum_{1 \leq i < j \leq t} N(c_i, c_j)$

- ⋮
- and $\binom{k}{k}$ times in $\sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k} N(c_{i_1}, c_{i_2}, \dots, c_{i_k})$

So in the end, x is counted $1 - k + \binom{k}{2} - \binom{k}{3} + \binom{k}{4} - \dots + (-1)^k \binom{k}{k}$

$$= \binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \dots + (-1)^k \binom{k}{k}$$

$$= \binom{k}{0} (-1)^0 + \binom{k}{1} (-1)^1 + \binom{k}{2} (-1)^2 + \dots + \binom{k}{k} (-1)^k = (-1+1)^k = 0$$

times.

So the left-hand side and right-hand side count each element the same number of times, which means they must be equal.

Example:

Find the number of non-negative integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 18$$

where $x_i \leq 7$ for all $i = 1, 2, 3, 4$.

Let S be the set of integer solutions of $x_1 + x_2 + x_3 + x_4 = 18$ with $0 \leq x_i$ for $i = 1, 2, 3, 4$.

$$\text{We learned: } |S| = \binom{18+4-1}{18} = \binom{18+4-1}{3} \quad (\text{easter eggs \& box dividers})$$

(We say that a solution $(x_1, x_2, x_3, x_4) \in S$ satisfies condition c_i if $x_i > 7$.
The answer to our question is now $N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4)$.

(What is $N(c_1)$? The number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 18$ with $0 \leq x_1, 0 \leq x_3, 0 \leq x_4$ and $8 \leq x_2$.

This is equal to the number of integer solutions to $y_1 + y_2 + y_3 + y_4 = 10$ with $0 \leq y_i$ for $i = 1, 2, 3, 4$.

$$\text{Hence } N(c_1) = \binom{10+4-1}{10} = \binom{13}{10}.$$

$$\text{Similarly, } N(c_2) = N(c_3) = N(c_4) = \binom{13}{10}.$$

(What is $N(c_1 c_2)$? The number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 18$ with $x_1 > 8, x_2 > 8, x_3 > 0$ and $x_4 > 0$.

This is equal to the number of integer solutions to $y_1 + y_2 + y_3 + y_4 = 2$ with $y_1 > 0, y_2 > 0, y_3 > 0$ and $y_4 > 0$

$$\text{Hence } N(c_1 c_2) = \binom{2+4-1}{2} = \binom{5}{2}$$

$$\text{Similarly, } N(c_1 c_3) = N(c_1 c_4) = N(c_2 c_3) = N(c_2 c_4) = N(c_3 c_4) = \binom{5}{2}$$

$$N(c_i c_j c_k) = 0 \text{ for } i < j < k \text{ since } 3 \cdot 8 = 24 > 18.$$

$$N(c_1 c_2 c_3 c_4) = 0 \text{ as well.}$$

$$\begin{aligned} \text{Now } N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4) &= N - \sum_{1 \leq i \leq 4} N(c_i) + \sum_{1 \leq i < j \leq 4} N(c_i c_j) - \sum_{1 \leq i < j < k \leq 4} N(c_i c_j c_k) + N(c_1 c_2 c_3 c_4) \\ &= \binom{21}{3} - 4 \cdot \binom{13}{10} + 6 \cdot \binom{5}{2} - 0 + 0 = 246 \end{aligned}$$

Example:

Determine the number of positive integers n where $1 \leq n \leq 100$ and n is not divisible by 2, 3 or 5.

$$\text{Then } S = \{1, 2, \dots, 100\}$$

c_1 = being divisible by 2

c_2 = being divisible by 3

c_3 = being divisible by 5

We must find: $N(\bar{c}_1 \bar{c}_2 \bar{c}_3)$.

$$\left. \begin{array}{l} N(c_1) = \lfloor \frac{100}{2} \rfloor = 50 \\ N(c_2) = \lfloor \frac{100}{3} \rfloor = 33 \\ N(c_3) = \lfloor \frac{100}{5} \rfloor = 20 \\ N(c_1 c_2) = \lfloor \frac{100}{6} \rfloor = 16 \\ N(c_2 c_3) = \lfloor \frac{100}{15} \rfloor = 6 \\ N(c_1 c_3) = \lfloor \frac{100}{10} \rfloor = 10 \\ N(c_1 c_2 c_3) = \lfloor \frac{100}{30} \rfloor = 3 \end{array} \right\}$$

explain $\lfloor \quad \rfloor$
and why this is correct

$$N(\bar{c}_1 \bar{c}_2 \bar{c}_3) = 100 - (50 + 33 + 20) + (16 + 6 + 10) - 3 = 26$$