Lesplan Maandag 1 Februari 2016 : Sets

"Definition": A <u>set</u> is a well-defined collection of objects. These objects are called <u>elements</u>, and are said to be <u>members</u> of the set.

Notation: The set with elements 1, 2, 3, 4 and 5 can be written as

$$\{1,2,3,4,5\} = \{1,1,2,3,2,4,5,2\}$$
$$= \{x \mid x \text{ is an integer and } 1 \le x \le 5\}$$
$$= \{\underbrace{x}_{\text{all } x} \underbrace{\mid}_{\text{such that}} \underbrace{x \text{ is an integer and } 0<x<6}_{\text{logical statement } p(x) \text{ true.}}\}$$

If we specify that we are working in the universe $\mathcal{U} = \{\text{integers}\}$ then

$$\{1,2,3,4,5\} = \{x \mid 1 \le x \le 5\}$$

Example: Universe: $\mathcal{U} = \{1,2,3,\dots\} = \{\text{positive integers}\}$

$$A = \{1,4,9,\dots,64,81\} = \{x^2 \mid x \in \mathcal{U} \wedge x^2 < 100\}$$
$$= \{x^2 \mid x \in \mathcal{U}, x^2 < 100\} = \{x \mid \exists y, y \in \mathcal{U}, x = y^2, y < 10\}$$
$$B = \{1,4,9,16\} = \{y^2 \mid y \in \mathcal{U}, y^2 < 20\} = \{y^2 \mid y \in \mathcal{U}, y^2 < 23\}$$
$$C = \{2,4,6,8,\dots\} = \{2k \mid k \in \mathcal{U}\} = \{k \mid k \in \mathcal{U}, k \text{ even}\}$$

Notation: "$64 \in A$" means that 64 is an element of A.

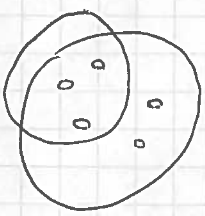
"$64 \notin B$" means that 64 is not an element of B.

Conventionally, sets are denoted by uppercase letters, and elements (if they need a name) by lowercase letters.

"$|A|$" means the number of elements of A $(=9)$ = <u>cardinality</u>

Notice that sets can also be infinite, such as C.

Definition: If C, D are sets from a universe $\mathcal{U}$, then we say that C is a subset of D (write $C \subseteq D$) if each element of C is also an element of D.



This can be expressed as "$C \subseteq D$", "$D \supseteq C$", "D contains C", "C is contained in D", "C is a subset of D"

If $C \subseteq D$ and D has an element d that is not an element of C, then C is called a <u>proper</u> subset of D. Notation $C \subsetneq D$.

The notation $C \subset D$ means $\subsetneq$ in the book, but many people use it to mean $\subseteq$. Please do <u>not</u> use this notation or explain what you mean by it

Remark: If C and D are sets in a universe $\mathcal{U}$:

- If $C \subsetneq D$, then $C \subseteq D$.
- If C and D are finite and $C \subseteq D$, then $|C| \leq |D|$
- If C and D are finite and $C \subsetneq D$, then $|C| < |D|$

Definition: If C and D are sets in a universe $\mathcal{U}$, then $C = D$ if and only if $C \subseteq D$ and $D \subseteq C$.

KAHOOT quiz

Logical statements: A, B sets from universe $\mathcal{U}$. Then

$$A \subseteq B \iff \forall x\, [x \in A \Rightarrow x \in B]$$

$$\begin{aligned} A \not\subseteq B &\iff \neg \forall x\, [x \in A \Rightarrow x \in B] \\ &\iff \exists x\, \neg [x \in A \Rightarrow x \in B] \\ &\iff \exists x\, \neg [x \in B \vee x \notin A] \\ &\iff \exists x\, [\neg(x \in B) \wedge \neg(x \notin A)] \\ &\iff \exists x\, [x \notin B \wedge x \in A] \iff \exists x\, [x \in A \wedge x \notin B] \end{aligned}$$

Remark: $A \subsetneq B \iff A \subseteq B \wedge A \neq B$

Theorem: Let $A, B, C \subseteq \mathcal{U}$

a) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

b) If $A \subsetneq B$ and $B \subseteq C$, then $A \subsetneq C$

c) If $A \subseteq B$ and $B \subsetneq C$, then $A \subsetneq C$

d) If $A \subsetneq B$ and $B \subsetneq C$, then $A \subsetneq C$

Proof: We will use the rule of universal generalization: let x be an arbitrarily chosen (but fixed once chosen) element of $\mathcal{U}$.

a) We want to prove (a): If $x \in A$, then $x \in C$.
So assume that $x \in A$. Then since $A \subseteq B$, we must have $x \in B$ (univ. sp.)
Since $x \in B$ and $B \subseteq C$, we have $x \in C$.
So if $x \in A$, then $x \in C$, which means that $A \subseteq C$.

b) Let $x \in \mathcal{U}$ be arbitrary. We want to prove that $A \subsetneq C$, so assume that $x \in A$. Since $A \subsetneq B$, we find that $x \in B$. As $x \in B$ and $B \subseteq C$, we see that $x \in C$. Hence $A \subseteq C$.
As $A \subsetneq B$, there exists $b \in B$ such that $b \notin A$. Since $b \in B$ and $B \subseteq C$, we find that $b \in C$.
So $A \subseteq C$ and we have an element $b \in C$ with $b \notin A$. So $A \subsetneq C$.

Definition: The empty set $\emptyset$ is the (unique) set without any elements. It is also denoted by $\{\}$

Remark: $|\emptyset| = 0$

$\{\emptyset\} \neq \emptyset$ since $|\{\emptyset\}| = 1$

$\{0\} \neq \emptyset$

$|\{\emptyset, \{\emptyset\}\}| = 2$

Theorem: For any universe $\mathcal{U}$ and $A \subseteq \mathcal{U}$:

- $\emptyset \subseteq A$
- If $A \neq \emptyset$, then $\emptyset \subsetneq A$

Proof: Suppose "$\emptyset \subseteq A$" is not true. Then there is an element $x \in \emptyset$ with $x \notin A$. But "$x \in \emptyset$" is impossible. Hence "$\emptyset \subseteq A$" must be true.

If $A \neq \emptyset$, then A has at least one element. So we can find $a \in A$. Since $a \notin \emptyset$ and $\emptyset \subseteq A$, we find that $\emptyset \subsetneq A$.

Example: $A = \{1,2,3,4,5\}$. How many subsets does A have?

1 of ~~ca~~ cardinality 0 : $\emptyset$

$\binom{5}{1}$ of cardinality 1 : $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$

$\binom{5}{2}$ 2

$\binom{5}{3}$ 3

$\binom{5}{4}$ 4

$\binom{5}{5}$ 5 : $\{1,2,3,4,5\}$

So $\binom{5}{0} + \binom{5}{1} + \dots + \binom{5}{5} = 2^5$ subsets.

Easier way to reach 2^5: To construct a subset we must choose, for each element, whether or not to include it. By rule of product, we can do this in 2^5 ways.

Definition: If $A \subseteq \mathcal{U}$ is a set in a universe, then $\mathcal{P}(A)$ is the set of all subsets of A. This is called the power set of A

Example: $A = \{1,2,3\}$

$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

Remark: If A is finite, $|\mathcal{P}(A)| = 2^{|A|}$

Some important sets:

$\mathbb{N} = \{$non-negative integers$\} = \{0,1,2,\dots\}$ (varies!)

$\mathbb{Z} = \{$integers$\} = \{0,1,-1,2,-2,3,-3,\dots\}$

$\mathbb{Z}^+ = \{$positive integers$\} = \{1,2,3,\dots\}$

$\mathbb{Q} = \{$rational numbers$\} = \{\frac{a}{b} \mid a,b\in\mathbb{Z}, b\neq 0\}$

$\mathbb{Q}^+ = \{$positive rational numbers$\} = \{\frac{a}{b} \mid a,b\in\mathbb{Z}^+\} = \{r\in\mathbb{Q} \mid r>0\}$

$\mathbb{Q}^* = \{$non-zero rational numbers$\} = \{r\in\mathbb{Q} \mid r\neq 0\}$

$\mathbb{R} = \{$real numbers$\}$

$\mathbb{R}^+ = \{$positive real numbers$\}$

$\mathbb{R}^* = \{$non-zero real numbers$\}$

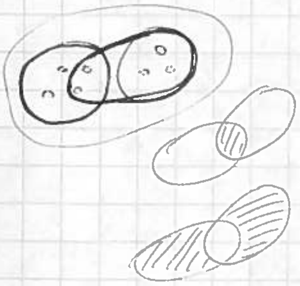
$\mathbb{C} = \{$complex numbers$\} = \{x+yi \mid x,y\in\mathbb{R}, i^2=-1\}$

$\mathbb{C}^* = \{$non-zero complex numbers$\}$

$\mathbb{Z}_n = \{0,1,2,\dots,n-1\} = \{x\in\mathbb{Z} \mid 0\leq x\leq n-1\}$ where $n\in\mathbb{Z}^+$.

For $a,b\in\mathbb{R}$ with $a<b$:

$[a,b] = \{x\in\mathbb{R} \mid a\leq x\leq b\}$ closed interval
$(a,b) = \{x\in\mathbb{R} \mid a< x< b\}$ open interval
$(a,b] = \{x\in\mathbb{R} \mid a< x\leq b\}$ } half-open interval
$[a,b) = \{x\in\mathbb{R} \mid a\leq x< b\}$



Definition: A,B sets in a universe $\mathcal{U}$.

$A\cup B = \{x \mid x\in A \vee x\in B\}$ = the union of A and B

$A\cap B = \{x \mid x\in A \wedge x\in B\}$ = the intersection of A and B

$A\Delta B = \{x \mid (x\in A \vee x\in B) \wedge \neg(x\in A \wedge x\in B)\}$

$= \{x \mid x\in A\cup B \wedge x\notin A\cap B\}$ = the symmetric difference of A and B

"$\cup$", "$\cap$" and "Δ" are binary operators on $\mathcal{P}(\mathcal{U})$:

they have two elements of $\mathcal{P}(\mathcal{U})$ as input and then give an output. Since the output is also an element of $\mathcal{P}(\mathcal{U})$, the binary operators are called "closed".

Example: $\mathcal{U} = \{1,2,\ldots,10\}$ $\quad A = \{1,2,3,4,5\}$ $\quad B = \{3,4,5,6,7\}$, $C = \{7,8,9\}$

$A \cap B = \{3,4,5\}$ $\qquad A \cup B = \{1,2,3,4,5,6,7\}$

$B \cap C = \{7\}$ $\qquad A \cap C = \emptyset$

$A \,\Delta\, B = \{1,2,6,7\}$ $\qquad A \cup C = \{1,2,3,4,5,7,8,9\}$

$A \,\Delta\, C = \{1,2,3,4,5,7,8,9\}$

Remark: For sets A, B in a universe $\mathcal{U}$:

$$A \cap B \subseteq A \subseteq A \cup B$$

Proof: Let $x \in \mathcal{U}$ be arbitrary. If $x \in A \cap B$, then $x \in A \wedge x \in B$, so $x \in A$.
Hence $A \cap B \subseteq A$.
If $x \in A$, then $x \in A \vee x \in B$, so $x \in A \cup B$. Hence $A \subseteq A \cup B$.

Definition: Let $A, B \subseteq \mathcal{U}$. We call A and B (mutually) disjoint if $A \cap B = \emptyset$

Theorem: Let $A, B \subseteq \mathcal{U}$. Then A and B are disjoint if and only if $A \cup B = A \,\Delta\, B$.

Proof: We need to prove "if" and "only if". Let's start with "only if":

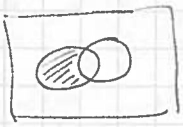
- Suppose that A and B are disjoint.
To prove that $A \cup B = A \Delta B$, we need to show "$A \cup B \subseteq A \Delta B$" and "$A \cup B \supseteq A \Delta B$".
Let $x \in \mathcal{U}$ be arbitrary. If $x \in A \cup B$, then $x \in A$ or $x \in B$ or both. Since A and B are disjoint, we have $A \cap B = \emptyset$ so $x \notin A \cap B$. As $x \in A \cup B$ and $x \notin A \cap B$, we find that $x \in A \Delta B$. Since this works for any $x \in \mathcal{U}$, we now know that $A \cup B \subseteq A \Delta B$.
Let $y \in \mathcal{U}$ be arbitrary. If $y \in A \Delta$ then $y \in A \cup B$ by definition.
Hence $A \Delta B \subseteq A \cup B$.
As $A \cup B \subseteq A \Delta B$ and $A \Delta B \subseteq A \cup B$, we find $A \cup B = A \Delta B$.

- We will prove the "if" direction by Modus Tollens.
Suppose that A and B are not disjoint. Then we can find $x \in A \cap B$.
Then $x \in A$ and $x \in B$, hence also $x \in A \cup B$.
But as $x \in A \cap B$, we find $x \notin A \Delta B$.
Therefore, $A \cup B \not\subseteq A \Delta B$ and $A \cup B \neq A \Delta B$.

Definition: For a set A in universe $\mathcal{U}$, the complement of A is:

$$\overline{A} = \mathcal{U} - A = \mathcal{U} \setminus A = \{x \mid x \in \mathcal{U} \wedge x \notin A\}$$

For $A, B \subseteq \mathcal{U}$, the relative complement of A in B is:



$$B - A = B \setminus A = \{x \mid x \in B \wedge x \notin A\}$$

Example: $\mathcal{U} = \{1,2,3,4,5,6,7\}$ $\qquad A = \{1,2,3,4\}$ $\quad B = \{2,6\}$