Lecture plan Mandag 15 februar 2016: Relations

Definition: pag 274

For sets A and B, the Cartesian product (= cross product) is defined by

$$A \times B = \{(a,b) \mid a \in A,\ b \in B\}$$

The elements of $A \times B$ are ordered pairs. They are exactly all combinations of elements of A and elements of B.

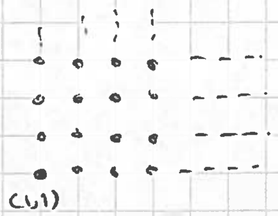
Example:

$A = \{1,2,3\}$ $B = \{1,x\}$

$A \times B = \{(1,1),(1,x),(2,1),(2,x),(3,1),(3,x)\}$

Example:

$\mathbb{Z}^+ \times \mathbb{Z}^+ = \{(a,b) \mid a \in \mathbb{Z}^+, b \in \mathbb{Z}^+\}$



Remark:

$A \times B$ and $B \times A$ are generally not equal. In fact, they are equal if and only if $A = B$.

Remark:

If A and B are finite, then $|A \times B| = |A| \cdot |B|$ by rule of product

Remark:

If $n \in \mathbb{Z}^+$ and $A_1, A_2, \ldots, A_n$ are sets, with $n \geq 3$, we can define
$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, \ldots, n\}$
The element $(a_1, a_2, \ldots, a_n)$ is called an n-tuple.

Theorem 1: pag 277

For any sets A, B and C in a universe $\mathcal{U}$:

a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$
d) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Proof:

We prove only part (a) here.
Suppose $x \in A \times (B \cap C)$. Then x has the form $x = (a,b)$ where $a \in A$ and $b \in B \cap C$. Since $a \in A$ and $b \in B$, $x \in A \times B$. Since $a \in A$ and $b \in C$, $x \in A \times C$. Hence $x \in (A \times B) \cap (A \times C)$.
This proves that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.
Now suppose $y \in (A \times B) \cap (A \times C)$. Then $y \in A \times B$ and $y \in A \times C$.
As $y \in A \times B$, y has the form $y = (a,b)$ with $a \in A$ and $b \in B$.
As $y = (a,b)$ and $y \in A \times C$, we find $b \in C$. Since $b \in B$ and $b \in C$, we have $b \in B \cap C$. Hence $y = (a,b) \in A \times (B \cap C)$.
This proves that $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.
Together, this proves (a).

Remark:

For any set A, $A \times \emptyset = \emptyset$ and $\emptyset \times A = \emptyset$.

Definition: For sets A and B, a (binary) relation from A to B is a subset $\mathcal{R} \subseteq A \times B$.
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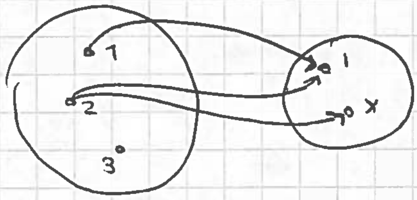
For a set A, a (binary) relation on A is a subset $\mathcal{R} \subseteq A \times A$.

Example: $A = \{1,2,3\}$ $\quad B = \{1, x\}$ $\quad \mathcal{R} = \{(1,1), (2,1), (2,x)\}$



Example:
$A = \{\text{students at NTNU}\}$
$B = \{\text{courses at NTNU}\}$
$\mathcal{R} = \{ (s,c) \mid s \in A, c \in B, \text{ student } s \text{ takes course } c \}$

Remark: For finite sets A and B with $|A| = m$ and $|B| = n$:

$$\#\{\text{relations from } A \text{ to } B\} = \#\{\text{subsets of } A \times B\}$$
$$= 2^{m \cdot n} \quad \text{since } |A \times B| = |A| \cdot |B| = mn$$

Likewise $\#\{\text{relations from } B \text{ to } A\} = 2^{n \cdot m} = 2^{mn}$.

If $\mathcal{R}$ is a relation from A to B, we can use $\mathcal{R}$ to construct a relation from B to A:

$\{ (b,a) \in B \times A \mid (a,b) \in \mathcal{R} \}$ is a relation from B to A.

We'll mainly discuss relations on some set A, so let us build a collection of standard examples

Examples:

1) $A_1 = \mathbb{Z}$, $\mathcal{R}_1 = \{(x,y) \in \mathbb{Z}^2 \mid x \leq y\}$
2) $A_2 = \mathbb{Z}$, $\mathcal{R}_2 = \{(x,y) \in \mathbb{Z}^2 \mid x < y\}$

3) $\mathcal{U}$ universe, $A_3 = \mathcal{P}(\mathcal{U})$, $\mathcal{R}_3 = \{ (X,Y) \in \mathcal{P}(\mathcal{U})^2 \mid X \subseteq Y\}$

4) $\mathcal{U}$ universe, $S \subsetneq \mathcal{U}$, $A_4 = \mathcal{P}(\mathcal{U})$

$\mathcal{R}_5 = \{(X,Y) \in \mathcal{P}(\mathcal{U})^2 \mid X \cap S = Y \cap S\}$

5) $A_5 = \mathbb{Z}^+$, $\mathcal{R}_5 = \{ (x,y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid x \mid y\}$

$= \{(x,y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid \exists c \in \mathbb{Z}^+, y = cx\}$

6) $A_6 = \{1,2,3,4,6,12\} = \{\text{positive integers that divide } 12\}$

$\mathcal{R}_6 = \{(x,y) \in A_6 \times A_6 \mid x \mid y\}$

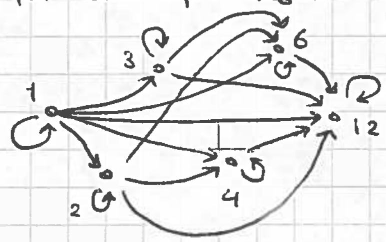
7) $A_7 = \mathbb{Z}$, $\mathcal{R}_7 = \{ (x,y) \in \mathbb{Z} \times \mathbb{Z} \mid 5 \mid x-y\}$

Notation: If R is a relation from A to B, we write:

aRb if $(a,b) \in R$

$a \not R b$ if $(a,b) \notin R$.

Drawing: We can represent a relation using a directed graph.
See for example R_6 on A_6:



We draw an arrow from x to y if and only if $(x,y) \in R$

These arrows are called directed edges
The points are called vertices

Properties that relations may satisfy:

Definition: A relation R on a set A is called reflexive if for all $x \in A$ we have $(x,x) \in R$

Examples: 1, 3, 4, 5, 6, 7 are reflexive, while 2 is not.

Counting: If A is a finite set with n elements, there are $2^{(n^2)}$ relations on A.
How many of those are reflexive?
A relation R on A is a subset of $A \times A$. To be reflexive, R has to contain the n elements of the form (x,x) with $x \in A$.
The other $n^2 - n$ elements of $A \times A$ may or may not be included in R.
By rule of product, this gives 2^{n^2-n} relations that are reflexive.

Definition: A relation R on a set A is called symmetric if for all $x,y \in A$ we have $(x,y) \in R \Rightarrow (y,x) \in R$

Examples: 4 and 7 are symmetric; 1, 2, 3, 5, and 6 are not

Remark: When drawing a graph for a relation: if (x,y) and (y,x) are both in R, we draw an undirected edge:

Counting: If A is finite and $|A| = n$, how many symmetric relations are there on A?
$A \times A$ has n elements of the form (x,x) with $x \in A$
and $n(n-1)$ elements of the form (x,y) with $x,y \in A$ and $x \neq y$.
These last elements can be paired up: (x,y) & (y,x).
There are $n(n-1)/2$ such pairs.
To have a symmetric relation in R, you must include the whole pair, or no elements of the pair.

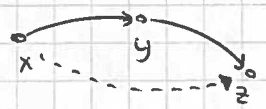
So we have $2^{\frac{n(n-1)}{2}}$ options to choose which pairs to include, and then 2^n options to choose which elements of the form (x,x) we include.

So there are $2^{\frac{n^2-n}{2}} \cdot 2^n = 2^{\frac{n^2+n}{2}}$ symmetric relations on A.

Counting: We can also count the symmetric reflexive relations on a set A with n elements.
In this case, we must include the n elements of the form (x,x).
So we can only choose which pairs to include.
This gives $2^{\frac{n^2-n}{2}}$ options.

Definition: A relation R on a set A is called transitive if

$$\forall x,y,z \in A, \left[(x,y)\in R \wedge (y,z)\in R \implies (x,z)\in R \right]$$



Examples: All the examples 1,2,3,4,5,6,7 are transitive.

Counting: There is no known general formula to count the number of transitive relations on a finite set A.

Definition: A relation R on a set A is called antisymmetric if $\forall x,y \in A$: if $(x,y)\in R$ and $(y,x)\in R$, then $x=y$.

Examples: 1, 2, 3, 5 and 6 are antisymmetric, while 4 and 7 are not

Remark: Relations can also be both symmetric <u>and</u> antisymmetric, or not symmetric and not antisymmetric. See Kahoot.

Counting: Let's count the anti-symmetric relations on a set A with n elements.

$A\times A$ contained elements (x,x) with $x\in A$
and elements of the form (x,y) with $x,y\in A$ and $x\neq y$.
The first type of element is irrelevant for anti-symmetry.
The elements of the second type come in pairs (x,y) & (y,x).
There are $n(n-1)/2$ such pairs. Of each pair, an anti-symmetric relation may contain at most one element.
This gives 3 options ((x,y), (y,x) or none).
So we have $3^{n(n-1)/2}$ ways to choose from the pairs,
and 2^n ways to choose from the elements of the first type.

This gives $2^n \cdot 3^{n(n-1)/2}$ possible anti-symmetric relations.