NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for
MA0301 Elementær diskret matematikk

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Date Signature

Problem 1

a) How many different arrangements are there for the letters in the word

$$ARRANGEMENT$$

b) How many of these arrangements have all the identical letters standing next to each other? (Such as the arrangement AAMRREETNNG for example.)

c) How many of the arrangements in (a) have no A's standing next to each other?

Solution 1 *a) There are* $\frac{11!}{2!2!2!2!1!1!1!} = \frac{11!}{16}$ *different arrangements for these letters.*

b) There are 7 *different letters, which we can arrange in* $7!$ *ways. Now add the double letters next to their twins to get all the arrangements where identical letters are standing next to each other. So we have* $7!$ *such arrangements.*

c) There are only two letters A. Let us count the number of arrangements where the letters A do stand next to each other. This comes down to counting the arrangements of $ARRNGEMENT$ *and then adding the other A next to the first one. There are* $\frac{10!}{2!2!2!1!1!1!1!} = \frac{10!}{8}$ *ways to arrange the letters of* $ARRNGEMENT$.

The number of arrangements of $ARRANGEMENT$ *where the letters A do not stand next to each other must now be equal to*

$$\frac{11!}{16} - \frac{10!}{8}.$$

Another way to solve this problem: first arrange the letters of $RRNGEMENT$, *which can be done in* $\frac{9!}{2!2!2!1!1!1!} = \frac{9!}{8}$ *ways. Now consider the 10 spots before, between and after these 9 letters. The letters A must occupy two different spots, which can be chosen in* $\binom{10}{2} = 45$ *ways. So there are* $45 \cdot \frac{9!}{8}$ *ways to arrange the letters of* $ARRANGEMENT$ *in such a way that the letters A do not stand next to each other.*

Notice that the two answers above are identical: $\frac{11!}{16} - \frac{10!}{8} = 11 \cdot 10 \cdot \frac{9!}{16} - 10 \cdot \frac{9!}{8} = \frac{11 \cdot 10}{2} \cdot \frac{9!}{8} - 10 \cdot \frac{9!}{8} = 55 \cdot \frac{9!}{8} - 10 \cdot \frac{9!}{8} = 45 \cdot \frac{9!}{8}$.

Problem 2

a) Prove the following set equality for sets A, B and C in a universe $\mathcal{U}$

$$(A \cup B) \cap \overline{\left((\overline{A} \cap \overline{B}) \cap C\right)} = (A \cup B) - C$$

You may use membership tables, the laws of set theory or element arguments.

b) Let X and Y be sets and let $f\colon X \to Y$ be a function. Let A and B be two subsets of X. Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$.

Solution 2 *a) To solve this problem, both membership tables and the laws of set theory will give a quick solution.*

Using the laws of set theory:

$$\begin{aligned}
(A \cup B) \cap \overline{\left((\overline{A} \cap \overline{B}) \cap C\right)} &= (A \cup B) \cap \left(\overline{\overline{(\overline{A} \cap \overline{B})}} \cup \overline{C}\right) \text{ by DeMorgan's Law} \\
&= (A \cup B) \cap \left((\overline{A} \cap \overline{B}) \cup \overline{C}\right) \text{ by the Law of Double Complement} \\
&= (A \cup B) \cap \left(\overline{(A \cup B)} \cup \overline{C}\right) \text{ by DeMorgan's Law} \\
&= \left((A \cup B) \cap \overline{(A \cup B)}\right) \cup \left((A \cup B) \cap \overline{C}\right) \text{ by the Distributive Law} \\
&= \emptyset \cup \left((A \cup B) \cap \overline{C}\right) \text{ by the Inverse Law} \\
&= (A \cup B) \cap \overline{C} \text{ by the Idempotent Law} \\
&= (A \cup B) - C
\end{aligned}$$

Using a membership table (or actually two, since it got too wide):

A	B	C	$A \cup B$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$	$\overline{(\overline{A} \cap \overline{B})}$	$(\overline{A} \cap \overline{B}) \cap C$	$\overline{\left((\overline{A} \cap \overline{B}) \cap C\right)}$
0	0	0	0	1	1	1	0	0	1
0	0	1	0	1	1	1	0	0	1
0	1	0	1	1	0	0	1	0	1
0	1	1	1	1	0	0	1	1	0
1	0	0	1	0	1	0	1	0	1
1	0	1	1	0	1	0	1	1	0
1	1	0	1	0	0	0	1	0	1
1	1	1	1	0	0	0	1	1	0

A	B	C	$(A \cup B) \cap \overline{\left((\overline{A} \cap \overline{B}) \cap C\right)}$	$(A \cup B) - C$
0	0	0	0	0
0	0	1	0	0
0	1	0	1	1
0	1	1	0	0
1	0	0	1	1
1	0	1	0	0
1	1	0	1	1
1	1	1	0	0

Since the sets $(A \cup B) \cap \overline{\left((\overline{A} \cap \overline{B}) \cap C\right)}$ and $(A \cup B) - C$ have the same membership tables, they must be equal.

b) Let $z \in f(A \cap B)$ be arbitrary. We will prove that $z \in f(A) \cap f(B)$.

As $z \in f(A \cap B)$, there must be an element $x \in A \cap B$ such that $z = f(x)$. As $x \in A \cap B$, we see that $x \in A$ and $x \in B$. As $x \in A$ and $z = f(x)$, we find that $z \in f(A)$. As $x \in B$ and $z = f(x)$, we find that $z \in f(B)$. Since $z \in f(A)$ and $z \in f(B)$, we may conclude that $z \in f(A) \cap f(B)$. Since z was an arbitrary element of $f(A \cap B)$, this proves that $f(A \cap B) \subseteq f(A) \cap f(B)$.

Problem 3

a) Is the function $f\colon \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2 + 2x + 1$ injective? Prove your answer.

b) Is the function given in (a) surjective? Prove your answer.

c) Give a restriction of the function in (a) that is injective.

Solution 3 *a) This function is not injective, since $f(0) = 1$ and $f(-2) = (-2)^2 + 2 \cdot (-2) + 1 = 1$.*

b) This function is not surjective, since for all $x \in \mathbb{Z}$ we have

$$f(x) = x^2 + 2x + 1 = (x + 1)^2 \geq 0.$$

So f will never assume any negative values, while its codomain does include negative values.

c) Take for example the restriction $f|_{\mathbb{N}}$ of f to $\mathbb{N}$. If $x, y \in \mathbb{N}$ with $x \neq y$, then we have $x < y$ or $x > y$. By swapping x and y if necessary, we may assume that $x < y$. Now $f|_{\mathbb{N}}(x) = x^2 + 2x + 1 < y^2 + 2y + 1 = f|_{\mathbb{N}}(y)$ since x and y are non-negative. Hence $f|_{\mathbb{N}}(x) \neq f|_{\mathbb{N}}(y)$. This proves that our restriction $f|_{\mathbb{N}}$ is injective.

Remark: you could also have solved this problem by taking a restriction to a very small set. Restrict f to the set $\{3\}$ for example. Since its domain has only one point, this restricted function will automatically be injective.

Problem 4

a) Construct a finite state machine (with input and output alphabet $\{0,1\}$) that recognises all strings that contain the string 10101 as a substring: the output of your Finite State Machine should be 1 when the input string contains 10101 as a substring, and 0 otherwise.

b) Give 6 different words in the language $\{11\}\{010\}^*\{3\} \cup \{1,22\}\{11\}^*$, where the alphabet is $\{0,1,2,3\}$. You should give at least one word that uses a letter 3 and at least one word that uses a letter 2.

Solution 4 *a) The Finite State Machine drawn in figure 1 recognises this language. There are also other valid solutions possible.*

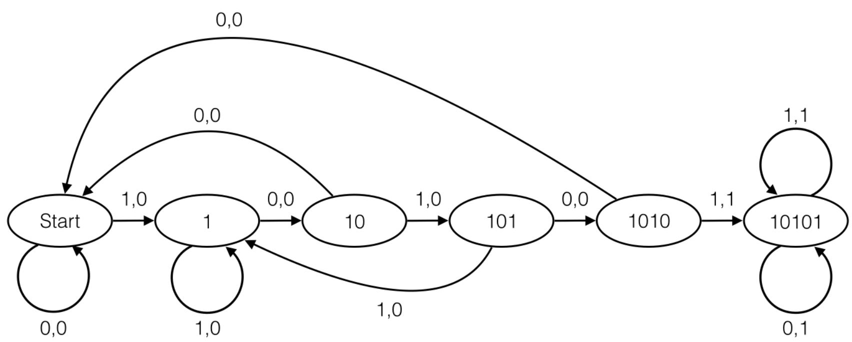

Figure 1: This figure belongs to Solution 4a

b) There are many possible answers to this question. For example, take the strings 113, 110103, 110100103, 1, 22, and 2211.

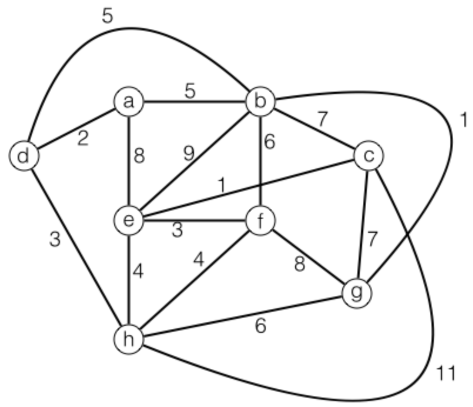

Figure 2: This figure belongs to Problem 5(a)

Problem 5

a) Find a minimal spanning tree for the graph in Figure 2 using Prim's algorithm. Start with vertex a and write down in which order you add your edges.

b) Study the graph in Figure 3 and determine whether or not is it planar. Prove your answer.

c) Does the graph in Figure 3 have a Hamilton cycle? Prove your answer.

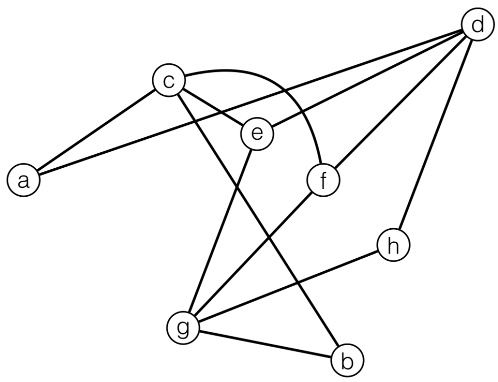

Figure 3: This figure belongs to Problem 5(b)

Solution 5 *a) We execute Prim's algorithm, starting at vertex a.*

For $i = 1$: $P_1 = \{a\}$, $N_1 = \{b, c, d, e, f, g, h\}$, $T = \emptyset$. Set $e_1 = \{a, d\}$.

For $i = 2$: $P_1 = \{a, d\}$, $N_1 = \{b, c, e, f, g, h\}$, $T = \{\{a, d\}\}$. Set $e_2 = \{d, h\}$.

For $i = 3$: $P_1 = \{a, d, h\}$, $N_1 = \{b, c, e, f, g\}$, $T = \{\{a, d\}, \{d, h\}\}$. Set $e_3 = \{h, e\}$. (There was another possible choice here, leading to another tree!)

For $i = 4$: $P_1 = \{a, d, h, e\}$, $N_1 = \{b, c, f, g\}$, $T = \{\{a, d\}, \{d, h\}, \{h, e\}\}$. Set $e_4 = \{e, c\}$.

For $i = 5$: $P_1 = \{a, d, h, e, c\}$, $N_1 = \{b, f, g\}$, $T = \{\{a, d\}, \{d, h\}, \{h, e\}, \{e, c\}\}$. Set $e_5 = \{e, f\}$.

For $i = 6$: $P_1 = \{a, d, h, e, c, f\}$, $N_1 = \{b, g\}$, $T = \{\{a, d\}, \{d, h\}, \{h, e\}, \{e, c\}, \{e, f\}\}$. Set $e_6 = \{a, b\}$. (There was another possible choice here, leading to another tree!)

For $i = 7$: $P_1 = \{a, d, h, e, c, f, b\}$, $N_1 = \{g\}$, $T = \{\{a, d\}, \{d, h\}, \{h, e\}, \{e, c\}, \{e, f\}, \{a, b\}\}$. Set $e_6 = \{b, g\}$.

Now $T = \{\{a, d\}, \{d, h\}, \{h, e\}, \{e, c\}, \{e, f\}, \{a, b\}, \{b, g\}\}$, which has weight 19.

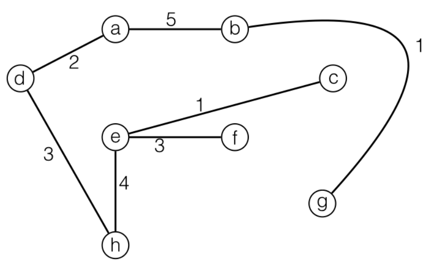

Figure 4: This figure belongs to Solution 5a

b) *This graph is planar. See Figure 5 for a planar embedding of this graph.*

c) *Figure 5 shows that our graph is bipartite, with 5 blue vertices and 3 red vertices. A Hamilton cycle must always pass from a red vertex to a blue one, and from a blue vertex to a red one. Since it must return to its starting vertex, the cycle must consist of the same number of blue vertices as red vertices. However, there are more blue than red vertices! Therefore, there cannot be a Hamilton cycle in this graph. (In fact, there can't even be a Hamilton path.)*

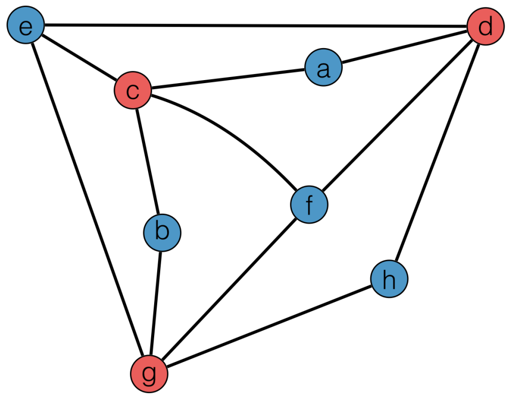

Figure 5: This figure belongs to Solutions 5b and 5c

Problem 6

a) Give the three defining properties of a partial ordering relation. You may give the names of these properties, or their definitions.

b) Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and define a relation $\mathcal{R}$ on S by setting

$$x\mathcal{R}y \Leftrightarrow x|y.$$

Prove that $\mathcal{R}$ is a partial ordering relation. (Recall: "$x|y$" is a notation for the statement "x divides y", or "$\frac{y}{x} \in \mathbb{Z}$" in other words.)

c) For the Hasse diagram in figure 6, identify all minimal, maximal, least and greatest elements (if they exist).

d) Prove that a partial ordering relation cannot have two distinct greatest elements.

Solution 6 *a) A partial ordering relation $\mathcal{R}$ on a set A has to be*

- *reflexive ($\forall a \in A : a\mathcal{R}a$)*
- *anti-symmetric ($\forall a, b \in A :$ if $a\mathcal{R}b$ and $b\mathcal{R}a$, then $a = b$), and*
- *transitive ($\forall a, b, c \in A :$ if $a\mathcal{R}b$ and $b\mathcal{R}c$, then $a\mathcal{R}c$).*

b) We need to prove that $\mathcal{R}$ is reflexive, anti-symmetric and transitive.

- *Reflexivity: If $x \in S$, then naturally we have $\frac{x}{x} = 1 \in \mathbb{Z}$. Hence $x|x$, so $x\mathcal{R}x$. This proves reflexivity.*
- *Anti-symmetry: If $x, y \in S$ with $x\mathcal{R}y$ and $y\mathcal{R}x$, then $x|y$ and $y|x$. Since we are dealing with positive integers, $x|y$ implies $x \leq y$ and $y|x$ implies $y \leq x$. Hence $x = y$. Therefore, the relation is symmetric.*
- *Transitivity: If $x, y, z \in S$ with $x\mathcal{R}y$ and $y\mathcal{R}z$, then we know that $x|y$ and $y|z$. This means that $\frac{y}{x}$ and $\frac{z}{y}$ are integers. Now $\frac{z}{x} = \frac{z}{y} \cdot \frac{y}{x}$, so as a product of two integers, it must also be an integer. Hence $x|z$ and so $x\mathcal{R}z$. This proves transitivity.*

Since all three properties are satisfied, $\mathcal{R}$ is a partial ordering relation.

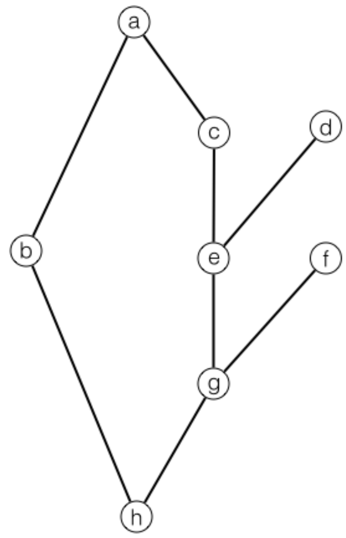

Figure 6: This figure belongs to Problem 6c

c) *Minimal elements: h.*

Maximal elements: a, d and f.

Least elements: h.

Greatest elements: none.

d) *Suppose that $\mathcal{R}$ is a partial ordering relation on a set A with two greatest elements x and y. Since x is a greatest element, we must have $y\mathcal{R}x$. Since y is a greatest element, we must have $x\mathcal{R}y$. We know that $\mathcal{R}$ is a partial ordering relation, so it must be anti-symmetric. As $x\mathcal{R}y$ and $y\mathcal{R}x$, anti-symmetry now tells us that $x = y$. Therefore, a partial ordering relation cannot have two distinct greatest elements.*