Answers to the Kahoot of February 29

Question 1

What properties does the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x - 14$ have?

We first study injectivity. Suppose x and y are two points in $\mathbb{R}$ where $f(x) = f(y)$. Then $3x - 14 = 3y - 14$, and hence we see that $3x = 3y$ (when we add 14 to both sides of the equation). Next, dividing by three shows us that $x = y$. So points in $\mathbb{R}$ can only have the same images if the points are equal. This proves that f is injective.

Now we study surjectivity. Thinking of the graph for this function gives us a strong feeling that this function is going to be surjective. So let $z \in \mathbb{R}$ be a point. We want to show that z is part of the range of f, so we want to find a point that is mapped to z.

So we are looking for a point $x \in \mathbb{R}$ such that $3x - 14 = z$. Solving this gives $3x = z + 14$, so $x = \frac{z+14}{3}$. And indeed, if we plug this in, we find $f(\frac{z+14}{3}) = 3 \cdot \frac{z+14}{3} - 14 = z + 14 - 14 = z$ so z is part of the range of f. This proves surjectivity.

Since the function f is both injective and surjective, it is bijective.

Question 2

What properties does the function $f\colon \mathbb{N} \to \mathbb{R}$ given by $f(x) = x^2 + 3x$ have?

We study injectivity first. Looking at the graph of this function implies that it increases on $\mathbb{N}$ so we will prove that the function is injective. Let x and y be points in $\mathbb{N}$ with $x \neq y$. Then either $x < y$ or $y < x$.

- If $x < y$: Since x and y are non-negative we have $x^2 < y^2$ and $3x < 3y$ so $x^2 + 3x < y^2 + 3y$. So $f(x) < f(y)$.
- If $y < x$: Since x and y are non-negative we have $y^2 < x^2$ and $3y < 3x$ so $y^2 + 3y < x^2 + 3x$. So $f(y) < f(x)$.

In both cases, we find that $f(x) \neq f(y)$ so f must be injective.

Next, we consider surjectivity. For any $n \in \mathbb{N}$, notice that $f(n) = n^2 + 3n \geq 0$ so the range can only contain non-negative numbers (and it doesn't contain all of them either). Therefore, the range is not all of $\mathbb{R}$ so f is not surjective.

In conclusion, this function is injective, but not surjective.

Question 3

What properties does the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} -x^2 & \text{if } x < 5 \\ x - 8 & \text{if } x \geq 5 \end{cases}$$

have?

Once again, it would be useful to draw a rough graph of this function to get an idea of what it looks like.

Let us consider injectivity. Since $f(1) = -1$ and $f(-1) = -1$ this function is not injective.

We move on to surjectivity. We consider the points of the codomain in two separate cases.

- If $b \in \mathbb{R}$ with $b < 0$, then $-b > 0$. So we can find the positive square root $a = \sqrt{-b}$. Then $-a$ is a negative real number, and $f(-a) = -(-a)^2 = -a^2 = -(-b) = b$ so b lies in the range of f.

- Now let $c \in \mathbb{R}$ with $c \geq 0$. Then $c + 8$ is a positive real number, and $c + 8 \geq 5$. Therefore, $f(c+8) = (c+8) - 8 = c$ so we see that c lies in the range of f.

This proves the surjectivity of f.

In conclusion, this function is surjective, but not injective.

Question 4

If $f\colon A \to B$ and $g\colon B \to C$ are functions and their composition $g \circ f\colon A \to C$ is injective, what do you know about f and g?

The function f has to be injective, but g does not need to be injective.

Let us first prove that f must be injective. Let x and y be points in A and suppose that $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$ and hence $(g \circ f)(x) = (g \circ f)(y)$. By injectivity of $g \circ f$, this implies that $x = y$. Therefore, the function f is injective. (We proved that if two points have the same image, then they are the same point.)

Now let us show thet g does not need to be injective. To do this, we need to give a counter-example. So set $A = \{a\}$, $B = \{b_1, b_2\}$ and $C = \{c\}$. We define $f\colon A \to B$ by $f(a) = b_1$ and $g\colon B \to C$ by $g(b_1) = c$ and $g(b_2) = c$. Then g is not injective.

The function $g \circ f$ is now given by $(g \circ f)(a) = c$. Since A has only one element, the function $g \circ f$ is automatically injective.

This shows that the injectivity of $g \circ f$ does not imply that g has to be injective.

Question 5

If $f\colon \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$, what is the set $f^{-1}\big([4, 9)\big)$?

The answer is $(-3, 2] \cup [2, 3)$.

We prove this by considering the points in the domain of f bit by bit:

- If $x \leq -3$, then $f(x) = x^2 \geq 9$. So then $f(x) \notin [4, 9)$ and hence $x \notin f^{-1}\big([4, 9)\big)$.

- If $-3 < x \leq -2$, then $4 \leq x^2 < 9$.So then $f(x) \in [4,9)$ and hence $x \in f^{-1}\big([4,9)\big)$.
- If $-2 < x < 2$, then $x^2 < 4$. So then $f(x) \notin [4,9)$ and hence $x \notin f^{-1}\big([4,9)\big)$.
- If $2 \leq x < 3$, then $4 \leq x^2 < 9$. So then $f(x) \in [4,9)$ and hence $x \in f^{-1}\big([4,9)\big)$.
- If $x \geq 3$, then $x^2 \geq 9$. So then $f(x) \notin [4,9)$ and hence $x \notin f^{-1}\big([4,9)\big)$.

This proves that $f^{-1}\big([4,9)\big) = (-3,2] \cup [2,3)$.

Question 6

Let $f\colon \{1,2,3,4,5,6\} \to \{x \in \mathbb{N} | 1 \leq x \leq 100\}$ be given by $f(x) = x^2$. How many extensions of f to $\{1,2,3,4,5,6,7,8,9,10\}$ are there?

To extend f to the set $\{1,2,3,4,5,6,7,8,9,10\}$, we need to give images to the points 7, 8, 9, and 10. For each of these points, we must choose an image from the set $\{x \in \mathbb{N} | 1 \leq x \leq 100\}$. So we have 100 possible images for 7, and 100 possible images for 8, and 100 possible images for 9, and 100 possible images for 10. This gives us 100^4 options to choose images for these four points.

Notice that the fact that f was originally defined by $f(x) = x^2$ is completely irrelevant for this problem.

Question 7

How many arrangements does the word $TALLAHASSEE$ have?

As a general theorem, we learned the number of arrangements of n objects of t different types, where we have

- k_1 identical objects of type 1
- k_2 identical objects of type 2
- k_3 identical objects of type 3
- ....
- k_t identical objects of type t

and $k_1 + k_2 + k_3 + \ldots + k_t = n$. This number was

$$\frac{n!}{k_1! k_2! k_3! \cdots k_t!}.$$

Applying this to the word $TALLAHASSEE$ gives the answer

$$\frac{11!}{3!2!2!2!1!1!} = \frac{11!}{3!2!2!2!}.$$

If you prefer to think the problem through rather than using the theorem, realise that the word $TA_1L_1L_2A_2HA_3S_1S_2E_1E_2$ has 11! different arrangements. However, if we remove the indices from the letters A, we see that words like

$A_1L_1L_2A_2HA_3S_1S_2E_2TE_1$,
$A_1L_1L_2A_3HA_2S_1S_2E_2TE_1$,
$A_2L_1L_2A_1HA_3S_1S_2E_2TE_1$,
$A_2L_1L_2A_3HA_1S_1S_2E_2TE_1$,
$A_3L_1L_2A_1HA_2S_1S_2E_2TE_1$ and
$A_3L_1L_2A_2HA_1S_1S_2E_2TE_1$

all become equal to $AL_1L_2AHAS_1S_2E_2TE_1$. So we apparently counted this new word 6 times. This number 6 corresponds exactly to the possible orders of A_1, A_2 and A_3.

If we now start removing the indices for the letters L, we see that we counted each new word 2 times. And removing the indices for the letters S shows that we once again counted everything twice. Finally, removing the indices for the letters E gives us another factor two.

This means that in removing the indices, there were apparently exactly $6 \cdot 2 \cdot 2 \cdot 2$ different arrangements of $TA_1L_1L_2A_2HA_3S_1S_2E_1E_2$ that correspond to the same arrangement of $TALLAHASSEE$. So the number of different arrangements of $TALLAHASSEE$ is

$$\frac{11!}{6 \cdot 2 \cdot 2 \cdot 2} = \frac{11!}{3!2!2!2!}$$

Question 8

How many times does the term $x^2y^3z^4$ appear in $(x+y+z+w)^9$?

Working out $(x+y+z+w)^9$ means working out the brackets in

$$(x+y+z+w) \cdot (x+y+z+w) \cdot (x+y+z+w) \cdot (x+y+z+w) \cdot (x+y+z+w) \cdot (x+y+z+w) \cdot (x+y+z+w) \cdot (x+y+z+w) \cdot (x+y+z+w)$$

To get a term of this expansion, we need to choose one item out of each set of brackets: either x or y or z or w. To end up with $x^2y^3z^4$, we must have chosen x twice, and y 3 times, and z 4 times, and no w. This can be done in any order.

So what we actually need to figure out, is in how many orders we could have picked 2 letters x, 3 letters y and 4 letters z. These are the arrangements of the word $xxyyyzzzz$. Using the theorem mentioned for problem 7, the answer will be

$$\frac{9!}{2!3!4!}.$$

Question 9

How many different terms of the form $x^a y^b z^d w^d$ (for all sorts of integers a, b, c and d) appear in $(x+y+z+w)^9$ once we have simplified the expression?

For example: $(x+y)^2 = xx + xy + yx + yy = x^2 + 2xy + y^2$ has 3 different terms left.

As stated earlier, working out $(x+y+z+w)^9$ means working out the brackets in

$$\begin{aligned}(x+y+z+w)\cdot(x+y+z+w)\cdot(x+y+z+w)\cdot(x+y+z+w)\cdot(x+y+z+w)\\ \cdot(x+y+z+w)\cdot(x+y+z+w)\cdot(x+y+z+w)\cdot(x+y+z+w)\end{aligned}$$

We need to figure out how many different terms we get. Any term will be a product of 9 letters, and these letters can be either x or y or z or w. So of our 9 letters, we need to choose how many are x, how many are y, how many are z and how many are w. Do we want 2 letters x, 2 letters y, 2 letters z and 3 letters w, for example? Or 9 letters x, 0 letters y, 0 letters z and 0 letters w? We need to count the total number of possibilities.

This is an Easter egg colouring problem: we have 9 easter eggs that we can give colours x, y, z or w. Each egg gets exactly one colour. This could be done in

$$\binom{9+(4-1)}{9} = \binom{12}{9} = \binom{12}{3}$$

different ways. So this is the number of truly different terms that we get.

Question 10

How many times does the term $x^2y^3z^4$ appear in $(3x+y-z+4w)^9$?

By the reasoning in question 8, we know that the term $(3x)^2y^3(-z)^4$ will appear $\frac{9!}{2!3!4!}$ times. Working this out, we get:

$$\frac{9!}{2!3!4!}\cdot(3x)^2y^3(-z)^4 = \frac{9!}{2!3!4!}\cdot 3^2(-1)^4x^2y^3z^4 = 3^2\cdot\frac{9!}{2!3!4!}\cdot x^2y^3z^4$$

So the term $x^2y^3z^4$ appears $3^2\cdot\frac{9!}{2!3!4!}$ times.