Lecture plan Onsdag 10 Februar 2016

1) Some more on induction: (§4.1)

Recall: Induction is used to prove a statement $S(n)$ (depending on some integer n) for all $n \in \mathbb{Z}^+$ (or all $n \geq 7$, or $n \geq -14$).

Basic step: $S(1)$ is true (or $S(7)$ or $S(-14)$)
Induction step: $\forall k \geq 1$, if $S(k)$ is true then $S(k+1)$ is true
↳ (or $k \geq 7$, or $k \geq -14$)

Example: $S(n)$: n can be expressed as a sum of 3's and 8's.
Prove $S(n)$ for all $n \geq 14$.

Basic step: $14 = 8+3+3$ so $S(14)$ is true.
Now let $k \geq 14$ be arbitrary and suppose as Induction Hypothesis that $S(k)$ is true.
Induction step: Assuming the IH, we will prove $S(k+1)$.

Since $S(k)$ is true, we can write $k = a \cdot 8 + b \cdot 3$ with $a, b \in \mathbb{N}$

- If $a \geq 1$, then $k+1 = a \cdot 8 + b \cdot 3 + 1 = (a-1) \cdot 8 + b \cdot 3 + 8 + 1$
$= (a-1) \cdot 8 + b \cdot 3 + 3 \cdot 3 = (a-1) \cdot 8 + (b+3) \cdot 3$
Since $a \geq 1$, we know that $a-1 \geq 0$. And $b+3 \geq 0$, so we have now written $k+1$ as a sum of 3's and 8's.
- If $a < 1$, then $a = 0$ and $k = b \cdot 3$. Since $k \geq 14$, we know that $b > 4$, so $b \geq 5$.
Now $k+1 = b \cdot 3 + 1 = (b-5) \cdot 3 + 5 \cdot 3 + 1 = (b-5) \cdot 3 + 16 = (b-5) \cdot 3 + 2 \cdot 8$
Since $b \geq 5$, we have $b-5 \geq 0$. Hence $k+1$ has now been written as a sum of 3's and 8's.

We see that $S(k+1)$ holds in any case!
By induction, we now know that $S(n)$ holds for all $n \geq 14$.

Theorem 2: Alternative form of the principle of induction
Let $S(n)$ be an open statement depending on the integer n, and let $n_0, n_1 \in \mathbb{Z}$ with $n_0 \leq n_1$
If the following are true:

a) $S(n_0), S(n_0+1), \ldots, S(n_1-1)$ and $S(n_1)$ are true

b) If $S(n_0), S(n_0+1), \ldots, S(k)$ are all true for some $k \in \mathbb{Z}$ with $k \geq n_1$, then $S(k+1)$ is also true.

Then $S(n)$ is true for all $n \geq n_0$.

Remark: The book says "$n_0, n_1 \in \mathbb{Z}^+$" and "$k \in \mathbb{Z}^+$" but this is not necessary.

Proof: The same idea used for theorem 1 works here
(Let $F = \{x \mid x \in \mathbb{Z},\ x \geq n_0,\ S(x) \text{ is false}\}$ and use a proof by contradiction to show that $F = \emptyset$. Use the fact that if $F \neq \emptyset$, it must have a smallest element.)

Example: $S(n)$: n can be expressed as a sum of 8's and 3's.
We prove $S(n)$ for all $n \geq 14$ using this alternative induction.

Take $n_0 = 14$ and $n_1 = 16$.

Basic step: $14 = 8+3+3$, $15 = 3+3+3+3+3$, $16 = 8+8$
so $S(14)$, $S(15)$ and $S(16)$ hold

Now let $k \in \mathbb{Z}$ with $k \geq 16$ and assume as an Induction Hyp. that $S(m)$ holds for all m such that $n_0 \leq m \leq k$.

Induction step: Assuming the induction hypothesis, we prove $S(k+1)$.
Notice that $k+1 \geq 16+1 = 17$ so $(k+1)-3 \geq 14$.
By the induction hypothesis, we can write $(k+1)-3$ as a sum of 3's and 8's.
But then $k+1$ is also a sum of 3's and 8's.
So $S(k+1)$ is true.

By induction, we know that $S(n)$ is true for all $n \geq 14$.

Example: We construct an integer sequence:

$a_0 := 1$
$a_1 := 2$
$a_2 := 3$
$a_n := a_{n-1} + a_{n-2} + a_{n-3}$ for all $n \in \mathbb{Z}$ with $n \geq 3$

$S(n)$: $a_n \leq 3^n$
We prove $S(n)$ for all $n \in \mathbb{Z}$ with $n \geq 0$.
Since calculating a_n requires information on a_{n-1}, a_{n-2} and a_{n-3}, we use the alternate induction, where $n_0 = 0$ and $n_1 = 2$.

Basic Step: $a_0 = 1 = 3^0 \leq 3^0$ so $S(0)$ holds
$a_1 = 2 \leq 3^1$ so $S(1)$ holds
$a_2 = 3 \leq 9 = 3^2$ so $S(2)$ holds.

Now let $k \in \mathbb{Z}$ with $k \geq 2$ be arbitrary and assume as Induction Hypothesis that $S(m)$ holds for all m such that $0 \leq m \leq k$.

Induction step: Assuming the IH, we prove $S(k+1)$:

$$a_{k+1} = a_k + a_{k-1} + a_{k-2} \overset{IH}{\leq} 3^k + 3^{k-1} + 3^{k-1} \leq 3^k + 3^k + 3^k = 3 \cdot 3^k = 3^{k+1}$$

so $S(k+1)$ holds.

By induction, we now know that $S(n)$ is true for all $n \geq 0$

Remark: With this new induction, we have a stronger induction hypothesis, which can make the induction step easier.

2) Recursive definitions : (§ 4.2)

Remark: In our last example, we defined a sequence by giving the first couple of terms and a "recipe" to calculate the other terms one by one.
This is called a recursive definition

Recursive	Direct
$a_0 = 1$ and $a_n = 2 \cdot a_{n-1}$ $\forall n \geq 1$	$a_n = 2^n$ $\forall n \geq 0$
To find a_{25}, I must first calculate $a_0, a_1, a_2, \dots, a_{24}$	To find a_{25}, I can just use the formula

(Note that I have defined the same sequence twice)

Remark: If your recursion says $a_{n+1} = 3a_n - 2a_{n-1} + a_{n-2} - a_{n-5}$, you should give the first 6 terms as starting terms.
(a_0, a_1, a_2, a_3, a_4 and a_5 will let you calculate a_6 and so on)

We could also have written $a_n = 3a_{n-1} - 2a_{n-2} + a_{n-3} - a_{n-6}$.
Or $a_{n+6} = 3a_{n+5} - 2a_{n+4} + a_{n+3} - a_n$

Similarly, instead of proving "$[S(k) \Rightarrow S(k+1)]$ $\forall k \geq 0$" you may prove "$[S(k-1) \Rightarrow S(k)]$ $\forall k \geq 1$" since it comes down to the same thing.

Example: The Fibonacci numbers have a nice recursive definition
- $F_0 = 0$, $F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$

$F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, ...

$$S(n): \sum_{i=0}^{n} F_i^2 = F_n \cdot F_{n+1}$$

We prove $S(n)$ for all $n \geq 0$ by induction

Basic Step: $\sum_{i=0}^{0} F_i^2 = F_0^2 = 0$ and $F_0 \cdot F_1 = 0 \cdot 1 = 0$ so $S(0)$ holds

Now let $k \in \mathbb{Z}$ with $k \geq 0$ be arbitrary and assume as an Induction Hypothesis that $S(k)$ holds.

Induction Step: Assuming the IH, we prove $S(k+1)$.

$$\sum_{i=0}^{k+1} F_i^2 = \left(\sum_{i=0}^{k} F_i^2\right) + F_{k+1}^2 \overset{IH}{=} F_k \cdot F_{k+1} + F_{k+1}^2$$
$$= F_{k+1}(F_k + F_{k+1}) = F_{k+1} \cdot F_{k+2}$$

so $S(k+1)$ holds.

By induction, we have now proved $S(n)$ for all $n \geq 0$.

Remark: As seen in the example above, recursive definitions and induction can work very well together.

Example: A careful definition of $p_1 \wedge p_2 \wedge \dots \wedge p_n$:

We learned the meaning of $p_1 \wedge p_2$ for statements p_1 & p_2.
If we have 3 statements, we may consider

$(p_1 \wedge p_2) \wedge p_3$ and $p_1 \wedge (p_2 \wedge p_3)$

which are equivalent by the Associative laws.
Since they are equivalent, we forget the brackets and write $p_1 \wedge p_2 \wedge p_3$ to mean any one of them.

It feels intuitive that we can do the same for $p_1 \wedge p_2 \wedge \dots \wedge p_n$.
Let's prove that this is right!

Definition: Given statements $p_1, p_2, \dots, p_n, p_{n+1}$ we define:

- $p_1 \wedge p_2$ as we know it
- $p_1 \wedge p_2 \wedge \dots \wedge p_{n+1} = (p_1 \wedge p_2 \wedge \dots \wedge p_n) \wedge p_{n+1}$ for $n \geq 2$.

Now set $S(n)$: For any statements $p_1, \dots, p_n$ and any $r \in \mathbb{Z}^+$ with $1 \leq r < n$ we have

$$(p_1 \wedge p_2 \wedge \dots \wedge p_r) \wedge (p_{r+1} \wedge \dots \wedge p_n)$$
$$\Updownarrow$$
$$p_1 \wedge p_2 \wedge \dots \wedge p_n$$

We will prove $S(n)$ for $n \geq 3$, by induction.

Basic Step: $n = 3$, and $r = 1$ or $r = 2$.

Associative law

For $r=1$: $p_1 \wedge (p_2 \wedge p_3) \Leftrightarrow (p_1 \wedge p_2) \wedge p_3 \Leftrightarrow p_1 \wedge p_2 \wedge p_3$

For $r=2$: $(p_1 \wedge p_2) \wedge p_3 \Leftrightarrow p_1 \wedge p_2 \wedge p_3$ by definition

So $S(3)$ holds.

Now let $k \in \mathbb{Z}$ with $k \geq 3$ be arbitrary and assume as an Induction Hypothesis that $S(m)$ is true for all m such that $3 \leq m \leq k$.

Induction Step: Assuming the IH, we prove $S(k+1)$.

If $r = k$, then $(p_1 \wedge \dots \wedge p_k) \wedge p_{k+1} \Leftrightarrow p_1 \wedge \dots \wedge p_{k+1}$ by def.
If $r < k$, then we have:

$$(p_1 \wedge \dots \wedge p_r) \wedge (p_{r+1} \wedge \dots \wedge p_{k+1})$$
$$\Leftrightarrow (p_1 \wedge \dots \wedge p_r) \wedge [(p_{r+1} \wedge \dots \wedge p_k) \wedge p_{k+1}] \quad \text{by def.}$$
$$\Leftrightarrow [(p_1 \wedge \dots \wedge p_r) \wedge (p_{r+1} \wedge \dots \wedge p_k)] \wedge p_{k+1} \quad \text{by Ass. law}$$
$$\Leftrightarrow [p_1 \wedge \dots \wedge p_k] \wedge p_{k+1} \quad \text{by IH}$$
$$\Leftrightarrow p_1 \wedge \dots \wedge p_k \wedge p_{k+1} \quad \text{by definition}$$

Hence $S(k+1)$ holds.

By induction, we've proved $S(n)$ for all $n \geq 3$.

Remark: We can similarly define $p_1 \vee p_2 \vee \dots \vee p_n$ and prove a similar statement.

Remark: We can also define $A_1 \cup A_2 \cup \dots \cup A_n$ like this, for sets $A_1, \dots, A_n$ in a universe $\mathcal{U}$:

- $A_1 \cup A_2$ is known
- $A_1 \cup A_2 \cup \dots \cup A_k \cup A_{k+1} := (A_1 \cup A_2 \cup \dots \cup A_k) \cup A_{k+1}$ for $k \geq 2$

Using another induction, we can show that the statement
$S(n)$: For any sets $A_1, A_2, \dots, A_n$ in any universe $\mathcal{U}$ and any $r \in \mathbb{Z}$ with $1 \leq r < n$, we have
$(A_1 \cup A_2 \cup \dots \cup A_r) \cup (A_{r+1} \cup \dots \cup A_n) = A_1 \cup A_2 \cup \dots \cup A_n$
is true for all $n \geq 3$.

Remark: But we have also defined $\bigcup_{i=1}^{n} A_i = \{x \in \mathcal{U} : \exists i, 1 \leq i \leq n, x \in A_i\}$.
Intuitively, this should be the same thing once again!
Let's prove it by induction.

$S(n)$: For all sets $A_1, A_2, \dots, A_n$ in any universe $\mathcal{U}$, we have $A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{i=1}^{n} A_i$

We will prove $S(n)$ for all $n \geq 2$.

Basic Step: If $x \in \mathcal{U}$ is arbitrary, then:
$x \in A_1 \cup A_2 \iff x \in A_1$ or $x \in A_2 \iff \exists i, [1 \leq i \leq 2 \wedge x \in A_i]$
$\iff x \in \bigcup_{i=1}^{2} A_i$

Hence $A_1 \cup A_2 = \bigcup_{i=1}^{2} A_i$

Now let $k \in \mathbb{Z}$ with $k \geq 2$ be arbitrary and assume as an Induction Hypothesis that $S(k)$ holds.

Induction Step: Assume the IH and let $x \in \mathcal{U}$ be arbitrary. Then:

$$x \in A_1 \cup \dots \cup A_{k+1} \iff x \in (A_1 \cup \dots \cup A_k) \cup A_{k+1}$$

$$\iff (x \in A_1 \cup \dots \cup A_k) \text{ or } x \in A_{k+1}$$

$$\overset{IH}{\iff} x \in \bigcup_{i=1}^{k} A_i \quad \text{or} \quad x \in A_{k+1}$$

$$\iff \left(\exists i, (1 \leq i \leq k) \wedge x \in A_i\right) \text{ or } x \in A_{k+1}$$

$$\iff \exists i, 1 \leq i \leq k+1 \wedge x \in A_i$$

$$\iff x \in \bigcup_{i=1}^{k+1} A_i$$

Hence $A_1 \cup \dots \cup A_{k+1} = \bigcup_{i=1}^{k+1} A_i$ so $S(k+1)$ holds.

By induction, $S(n)$ is now true for all $n \geq 2$.