Lecture plan Mandag 8 Februar 2016 : §4.1 Induction

- Introduce topic: last week an $a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow a$ proof.
  This week a method to prove statements $p(n)$ for all $n \in \mathbb{Z}^+$

<u>Remark:</u> $\mathbb{Z}^+ = \{x \in \mathbb{Z} \mid x > 0\} = \{x \in \mathbb{Z} \mid x \geq 1\}$
$\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$
$\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ } cannot be written like using $\geq$.

<u>The well-ordering principle:</u> Every non-empty subset $S \subseteq \mathbb{Z}^+$ has a smallest element.
(we do not prove this here)
(Other way to say this: $\mathbb{Z}^+$ is well-ordered)

<u>Remark:</u> $\mathbb{Q}^+$ and $\mathbb{R}^+$ are not well-ordered:
the subsets $\mathbb{Q}^+$ and $\{x \in \mathbb{Q}^+ \mid x > 4\}$ and $\{x \in \mathbb{Q} : x > \pi\}$ have no smallest elements. These are subsets of $\mathbb{Q}^+$ and $\mathbb{R}$
(Some subsets, like $\{x \in \mathbb{Q}^+ \mid x \geq 7\}$ do have a smallest element though)

<u>Theorem 1:</u> The principle of Mathematical Induction (page 208)
Let $S(n)$ be an open mathematical statement depending on the variable $n \in \mathbb{Z}$. If
a) $S(1)$ is true, and
b) whenever $S(k)$ holds (for some particular, arbitrarily chosen $k \in \mathbb{Z}^+$) we can prove that $S(k+1)$ is also true,

then $S(n)$ is true for all $n \in \mathbb{Z}^+$.

<u>Proof:</u> Let $F = \{t \in \mathbb{Z}^+ \mid S(t) \text{ is false}\}$. We want to show that $F = \emptyset$ using a proof by contradiction.
So suppose that $F \neq \emptyset$. Then F is a non-empty subset of $\mathbb{Z}^+$, so F has a smallest element m (by the well-ordering principle).
Since $S(1)$ is true, $1 \notin F$, so $m \neq 1$.
Now $m > 1$ so $m-1 \in \mathbb{Z}^+$. Since m was the smallest element of F, $m-1 \notin F$. Hence $S(m-1)$ is true.
By (b), we see that $S(m)$ is true, so $m \notin F$. Contradiction!
Hence, $F = \emptyset$, so $S(n)$ is true for all $n \in \mathbb{Z}^+$

<u>Logical statement:</u> $\left[S(1) \wedge [\forall k \geq 1 \, [S(k) \Rightarrow S(k+1)]] \right] \Rightarrow \forall n \geq 1 \, [S(n)]$

<u>Example:</u> Prove that for all $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

So $S(n)$ is the statement: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

<u>Basic step:</u> verify $S(1)$
$\sum_{i=1}^{1} i = 1$ and $\frac{1(1+1)}{2} = 1$ so $S(1)$ is true.

many other proofs exist

Let $k \in \mathbb{Z}^+$ be arbitrary and assume that the following <u>Induction Hypothesis</u> holds: $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

<u>Induction step:</u> prove that if $S(k)$ is true, then so is $S(k+1)$

$$\sum_{i=1}^{k+1} i = 1+2+3+\dots+k+(k+1) = \left(\sum_{i=1}^{k} i\right) + (k+1) \overset{IH}{=} \frac{k(k+1)}{2} + (k+1)$$
$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)}{2}(k+2) = \frac{(k+1)((k+1)+1)}{2}$$

So $S(k+1)$ is true.

By induction, $S(n)$ must be true for all $n \in \mathbb{Z}^+$

Example: Prove that for all $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ $(S(n))$

Basic step: $\sum_{i=1}^{1} i^2 = 1$ and $\frac{1(1+1)(2\cdot 1+1)}{6} = 1$ so $S(1)$ is true.

Let $k \in \mathbb{Z}^+$ be arbitrary and suppose that the following Induction Hypothesis holds:
$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Induction step: Under assumption of this induction hypothesis we find

$$\sum_{i=1}^{k+1} i^2 = \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2 \overset{IH}{=} \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$
$$= \frac{(k+1)\left(k(2k+1) + 6(k+1)\right)}{6}$$
$$= \frac{(k+1)(2k^2+k+6k+6)}{6} = \frac{(k+1)(2k^2+7k+6)}{6}$$
$$= \frac{(k+1)(k+2)(2k+1)}{6}$$

$$\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$
$$= \frac{(k+1)(2k^2+4k+3k+6)}{6}$$

Hence $S(k+1)$ is true.

By induction, $S(n)$ is true for all $n \in \mathbb{Z}^+$.

Wrong example: All students have the same age

Precise statement: In any group of n students $(n \in \mathbb{Z}^+)$, all students have the same age.

"Proof" by induction:

Basic step: If I have a group of 1 student, all students in this group have the same age.

Let $k \in \mathbb{Z}^+$ be arbitrary and assume that the following Induction Hypothesis holds: In any set of k students, all students have the same age.

Induction step: Assume the IH and let S be a set of $k+1$ students. Let $x \in S$ be a student. Then $S \setminus \{x\}$ is a set of k students, so all have the same age.
Let $y \in S$ another student. Then all students in $S \setminus \{y\}$ also ha the same age, so x has the same age as all t others.
Hence all students in S have the same age.

What is wrong here?

Other warning: Do not forget to chec the bas step. Without it, you migh be proving nonsense. See example 6.

Remark: The formulas $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{2}$ that we have proved are worth remembering.

They might speed up a program:

compare multiplication tables

```
begin
    sum := 0
    for i := 1 to n do
        sum = sum + i^2
end
```

n additions
n multiplications

```
begin
    sum := n*(n+1)*(2n+1)/6
end
```

2 additions
3 multiplications
1 division

And they can often be used while calculating other things:

Example: There are 900 3-digit integers.
A palindrome is an integer that remains the same if you reverse the order of its digits.
What is the sum of all 3-digit palindromes?

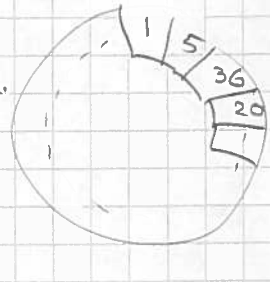
Form: aba with $1 \le a \le 9$ and $0 \le b \le 9$.
$aba = 100a + 10b + a = 101a + 10b$

By rule of product there are 90 3-digit palindromes
Let's calculate their sum:

$$\sum_{a=1}^{9}\left(\sum_{b=0}^{9} aba\right) = \sum_{a=1}^{9}\sum_{b=0}^{9}(101a + 10b)$$
$$= \sum_{a=1}^{9}\left(\sum_{b=0}^{9} 101a\right) + \sum_{a=1}^{9}\sum_{b=0}^{9} 10b$$
$$= \sum_{a=1}^{9} 10 \cdot 101 \cdot a + \sum_{a=1}^{9} 10 \cdot \sum_{b=0}^{9} b$$
$$= 10 \cdot 101 \cdot \sum_{a=1}^{9} a + 10 \cdot \sum_{a=1}^{9}\sum_{b=1}^{9} b$$
$$= 10 \cdot 101 \cdot \frac{9 \cdot 10}{2} + 10 \cdot \sum_{a=1}^{9} \frac{9 \cdot 10}{2}$$
$$= 10 \cdot 101 \cdot 45 + 10 \cdot 9 \cdot 45 = 49500$$

Example: (book does 55)

Wheel of fortune with numbers 1 up to 36, randomly distributed.
Show that there are 3 consecutive numbers with su 56 or more.
We prove this by contradiction, so assume it's not true.
Label the numbers n the wheel $x_1, x_2, \ldots, x_{36}$
Then $x_1 + x_2 + x_3 \le 55$, $x_2 + x_3 + x_4 \le 55$, ...,
$x_{34} + x_{35} + x_{36} \le 55$, $x_{35} + x_{36} + x_1 \le 55$, $x_{36} + x_1 + x_2 \le 55$.



Add all this:
$$3 \sum_{i=1}^{36} x_i \le 36 \cdot 55$$
$$\parallel$$
$$3 \cdot \frac{36 \cdot 37}{2}$$

But then $\frac{3 \cdot 37}{2} \le 55$, which is false. So the statement must be true!

Back to induction!

Example: Let $S(n)$ be the statement "$4n < (n^2-7)$"
Prove that $S(n)$ is true for all $n \in \mathbb{Z}^+$ with $n \geq 6$

We can still use induction here; we only need to change our basic step. Use $n=6$ as a basic step now!

Basic step: $4 \cdot 6 = 24$ and $(6^2-7) = 36-7 = 29$ so $S(6)$ is true.

Let $k \in \mathbb{Z}^+$ with $k \geq 6$ be arbitrary, and suppose that the following *Induction Hypothesis* holds: $S(k)$ is true (so $4k < k^2-7$)

Induction step: Assume the Induction Hypothesis. We need to prove that $4(k+1) < (k+1)^2-7$.

$(k+1)^2-7$
$= k^2+2k+1-7$

$4(k+1) = 4k+4 < k^2-7+4 = k^2+4-7$

$\leq k^2+2k+1-7 = (k+1)^2-7$

since $k \geq 6$, we have $4 \leq 2k+1$

By induction, $S(n)$ now holds for all $n \in \mathbb{Z}^+$ with $n \geq 6$

Remark: While the above can be proved by induction, it is not really efficient.
Shorter: Let $n \in \mathbb{Z}^+$ with $n \geq 6$.
Then $n^2-7 \geq 6n-7 = 4n+2n-7 \geq 4n+12-7 = 4n+5 > 4n$.
So if you have to prove some statement $S(n)$ for all $n \in \mathbb{Z}^+$, think whether or not induction seems wise.

Example: Sum of odd numbers
$1 = 1$
$1+3 = 4$
$1+3+5 = 9$
$1+3+5+7 = 16$

Can we prove that $\sum_{i=1}^{n}(2i-1) = n^2$? (for all $n \in \mathbb{Z}^+$)

Basic Step: $\sum_{i=1}^{1}(2i-1) = 2 \cdot 1 - 1 = 1$ and $1^2 = 1$ so the formula is true for $n=1$

Now let $k \in \mathbb{Z}^+$ be arbitrary and assume the following IH: $\sum_{i=1}^{k}(2i-1) = k^2$

Induction step: Assuming the Induction Hypothesis, we must prove that $\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$.

We have $\sum_{i=1}^{k+1}(2i-1) = \left(\sum_{i=1}^{k}(2i-1)\right) + (2(k+1)-1)$

$\overset{IH}{=} k^2+2k+2-1 = k^2+2k+1 = (k+1)^2$

By induction, we now know that $\sum_{i=1}^{n}(2i-1) = n^2$ for all $n \in \mathbb{Z}^+$

Example: For $n \in \mathbb{N}$, let $A_n \subseteq \mathbb{R}$ with $|A_n| = 2^n$ and list the elements of A_n in ascending order.

Claim: For $n \in \mathbb{N}$: For all A_n as above and $r \in \mathbb{R}$, we can determine whether or not $r \in A_n$ using at most $n+1$ comparisons to elements of A_n.

Basic step: If $n = 0$, then $|A_n| = 2^0 = 1$ so comparing r to the unique element of A_n is enough to see if $r \in A_n$.
So we need $1 = n+1$ comparison

Let $k \in \mathbb{N}$ and assume that the following IH holds: for any set $A_k \subseteq \mathbb{R}$ with $|A_k| = 2^k$ and a list of its elements in ascending order, we can determine if $r \in A_n$ with at most $k+1$ comparisons.

Induction step: Assume the IH. Let $A_{k+1} \subseteq \mathbb{R}$ with $|A_{k+1}| = 2^{k+1} = 2 \cdot 2^k$ and list its elements in ascending order:
$x_1 < x_2 < x_3 < \dots < x_{2^k} < y_1 < y_2 < \dots < y_{2^k}$

Define $B = \{x_1, \dots, x_{2^k}\}$ and $C = \{y_1, \dots, y_{2^k}\}$. Then $|B| = |C| = 2^k$, and $A_{k+1} = B \cup C$.
Compare r to y_1.

- If $r < y_1$, then $r \notin C$ and we need at most $k+1$ comparisons to see if $r \in B$ or not, by IH.
- If $r \geq y_1$, then $r \notin B$ and we need at most $k+1$ comparisons to see if $r \in C$ or not (by IH).

Hence in at most $1 + (k+1) = (k+1)+1$ comparisons to elements of A_{k+1}, we know if $r \in A_{k+1}$.

By induction, this proves the claim.

Remark: Induction can be very useful to check and prove that a bit of programming (using while loops for examp) does what it is supposed to do.

Example: Our program gets input $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $n \in \mathbb{N}$, and should compute $x(y^n)$.

```
while n ≠ 0 do
    begin
        x := x * y
        n := n-1
    end
answer := x
```

$S(n)$: the program correctly computes $x(y^n)$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}$ and the value n.

Basic step: If $n = 0$, the program skips the "begin... end" part and giv x. As $x(y^0)$ x, the program is correct here.

Let $k \in \mathbb{N}$ and assume that the induction hypothesis $S(k)$ holds.

Induction Step: We try to see if $S(k+1)$ holds, assuming the H.
Since $k \in \mathbb{N}$, we know $k+1 \geq 1$ so the "begin... end" part is not skipped.
After one round of this, we get a new value $x_1 = x * y$ instead of x, and a new value $n_1 = n-1 = (k+1)-1 = k$ instead of n.
By IH, the program now correctly compute $x_1 \cdot (y^{n_1})$, which is $(x*y) \cdot (y^k) = x(y^{k+1})$. So $S(k+1)$ holds.

By induction, this program gives the right answers.