

Lecture plan Wednesday March 30 : Graphs c

Recall:

Theorem 5:
page 595

A graph G is non-planar if and only if it contains a subgraph which is homeomorphic to K_5 or $K_{3,3}$.

Theorem 6:
page 597

Let $G = (V, E)$ be a connected planar (multi)-graph, with $|V| = v$ and $|E| = e$. Let r be the number of regions in the plane given by a planar embedding of G . Then $v - e + r = 2$.

Corollary:
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Let $G = (V, E)$ be a loop-free connected planar graph with $|V| = v$, $|E| = e$ and r regions, where $e \geq 2$. Then $3r \leq 2e$ and $e \leq 3v - 6$

Proof:

G is loop-free, not a multigraph, and has at least 3 edges. The boundary of each region of a planar embedding of G must now contain at least 3 edges, so each region has degree ≥ 3 (this also holds for the infinite region).

We have $2e = 2|E| =$ the sum of the degrees of all the regions $\geq 3r$ since any edge is part of the boundary of 2 regions (or twice part of the boundary of 1 region)

By Theorem 6, we now find $2 = v - e + r \leq v - e + (\frac{2}{3})e = v - (\frac{1}{3})e$.

This implies that $6 \leq 3v - e$, so $e \leq 3v - 6$

Remark:

If $G = (V, E)$ is a loop-free and connected graph with $|E| = e \geq 2$:

$$G \text{ planar} \Rightarrow 2e \geq 3r \text{ and } e \leq 3v - 6$$

$$\begin{array}{l} 2e < 3r \\ \text{or} \\ e > 3v - 6 \end{array} \Rightarrow G \text{ not planar}$$

Remark:

K_5 is loop-free and connected and $|E| = 10 \geq 2$. $|V| = 5$, and so $3v - 6 = 15 - 6 = 9 < 10 = e$. Hence K_5 is not planar.

Remark:

$K_{3,3}$ is loop-free and connected, and $|E| = 9 \geq 2$. $|V| = 6$ and hence $3v - 6 = 18 - 6 = 12 > 9 = e$ We cannot conclude anything from this calculation.

But $K_{3,3}$ is bipartite, and hence it cannot contain odd cycles. In particular, it cannot contain 3-cycles. The lowest possible degree of any region in any planar embedding of $K_{3,3}$ would therefore be 4. So $2e =$ the sum of the degrees of all the regions $\geq 4r$, if $K_{3,3}$ were planar.

By Theorem 6, we know that $v - e + r = 2$ and so $r = 2 - v + e = 2 - 6 + 9 = 5$. Hence $4r = 20$ and $2e = 18$.

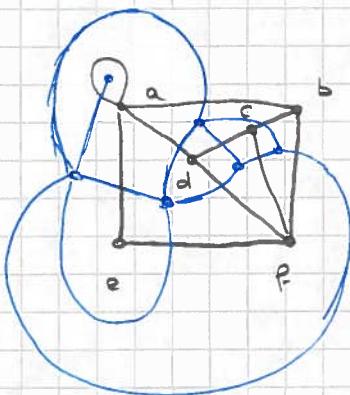
This shows that $K_{3,3}$ is not planar.

Definition:
see page 601

Let $G = (V, E)$ be a planar (multi)graph with a planar embedding. Then we can construct the dual relative to this embedding as follows :

- Call this dual graph $G^d = (V^d, E^d)$
- The vertices of G^d will be the regions of the embedding of G .
- For every edge $e \in E$, draw an edge in G^d that connects the regions adjacent to e .
(So if e does not separate two different regions in the embedding of G , then we create a loop in G^d)

Example:

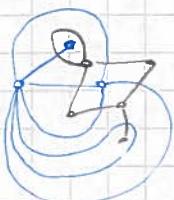


G is the original graph in a planar embedding.

The blue drawing is its dual

Remarks:

- 1) Any edge in G corresponds with an edge in G^d , and conversely.
- 2) A vertex of degree 2 in G yields a pair of edges in G^d that connect the same 2 vertices. (See e in the example)
So even if G is a graph, a dual of G might be a multigraph.



- 3) If G has a loop, and this loop has no other vertices in its enclosed area in the embedding, then the region in this loop turns into a pendant vertex of G^d .

Conversely, if G has a pendant vertex, this creates a loop in G^d .

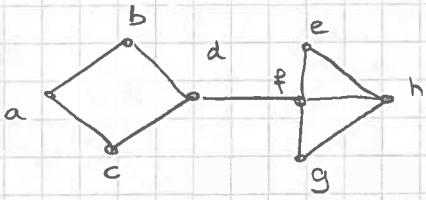
- 4) The degree of a vertex in G^d is equal to the degree of the corresponding region in G .
- 5) Different embeddings of a graph G may give different duals, and they do not have to be isomorphic to each other.

Definition:

Let $G = (V, E)$ be an undirected (multi)graph. A cut-set of G is a subset $E' \subseteq E$ such that

- Removing E' from G increases the number of connected components: $k(G - E') > k(G)$
- Removing only part of E' from G does not increase the number of connected components: If $E'' \subsetneq E'$, then $k(G - E'') = k(G)$

Example:



There are many different cut-sets, such as

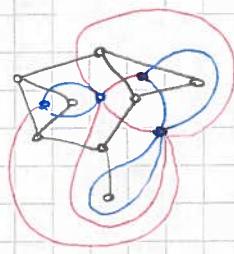
- $\{\{a,b\}, \{a,c\}\}$
 - $\{\{b,d\}, \{c,d\}\}$
 - $\{\{d,f\}\}$
 - $\{\{a,c\}, \{b,d\}\}$
- and so on ...

Definition:

A cut-set consisting of only one edge is called a bridge.

Remark:

Dual graphs connect cut-sets and cycles: Let G be a planar graph and G^d a dual for G .

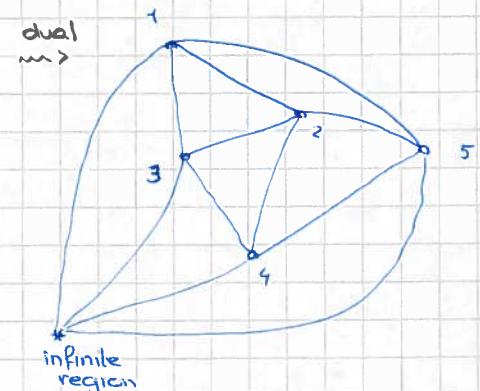
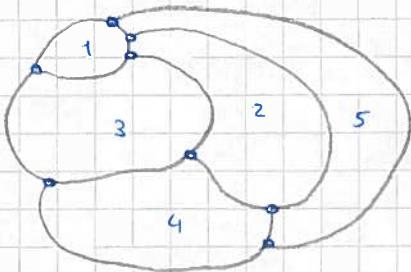


- 1) Cycles of $n (\geq 3)$ edges in G correspond with cut-sets of n edges in G^d
- 2) A loop in G corresponds with a one-edge cut-set in G^d
- 3) A one-edge cut-set in G corresponds with a loop in G^d
- 4) A two-edge cut-set in G corresponds with a two-edge circuit in G^d
- 5) If G is a planar multigraph, then each two-edge circuit in G determines a two-edge cut-set in G^d

Example:

Duals can be used to translate one question about graphs to another question (which is perhaps easier to visualize or think about).

Take for example the "mapmaker's problem": we want to colour countries on a map so that no neighbours have the same colour.



The corresponding problem in this dual graph means colouring the points (except the point corresponding to the infinite region) in such a way that no two adjacent points have the same colour.

Definition:
page 608

If $G = (V, E)$ is a (multi)graph with $|V| \geq 3$, a Hamilton cycle in G is a cycle that contains every vertex in V .
A Hamilton path is a path that visits every vertex in V .

Remark:

If you remove one edge from a Hamilton cycle, it becomes a Hamilton path.

Remark:

There is no general theorem that characterizes when Hamilton paths and cycles do or do not exist. There are many partial theorems though, that give either sufficient or necessary conditions.

Example:

The hypercube Q_3 has a Hamilton cycle:

$$000 \rightarrow 100 \rightarrow 110 \rightarrow 010 \rightarrow 011 \rightarrow 111 \rightarrow 101 \rightarrow 001 \rightarrow 000$$

Removing one edge will give a Hamilton path.

If we list the successive vertices in a Hamilton path in Q_3 , we get a list of binary numbers, up to 3 digits, where each number differs from the next number in the list in at most one digit. Such a list is called a Gray code.

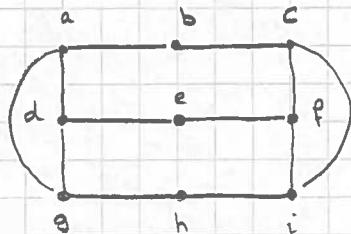
Hamilton cycles exist in all Q_n for $n \geq 2$.

Remark:

How can we try to find Hamilton cycles? Or prove it when there are none?

- 1) If G has a Hamilton cycle, then $\deg(v) \geq 2 \quad \forall v \in V$.
- 2) If $a \in V$ and $\deg(a)=2$ and G has a Hamilton cycle, then the two edges incident with a must be part of this Hamilton cycle.
- 3) If $a \in V$ and $\deg(a) > 2$ and we're trying to build a Hamilton cycle: once we pass through vertex a , (using two edges incident with a), we can no longer use any other edges incident with a .
- 4) If we are building a Hamilton cycle for G , it cannot contain a cycle for a subgraph of G (unless it contains all the vertices of G already)

Example:



There is a Hamilton-path:
 $a \rightarrow b \rightarrow c \rightarrow f \rightarrow e \rightarrow d \rightarrow g \rightarrow h \rightarrow i$

There is no Hamilton cycle:

If there were a Hamilton cycle, it would have to pass through b , e and h so it should include the pieces $a-b-c$, $d-e-f$ and $g-h-i$.

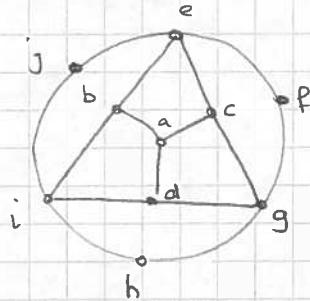
But we cannot patch these pieces together: each stretch sends us from $\{a,d,g\}$ to $\{c,f,i\}$ or back, and as we only have 3 pieces we must start in $\{a,d,g\}$ and end in $\{c,f,i\}$ or the other way around. As there are no edges between $\{a,d,g\}$ and $\{c,f,i\}$, we can never close the cycle.

Remark:



If G is a bipartite graph (with m red points and n blue points and no edges between 2 red points or between 2 blue points) and $m \neq n$, then G has no Hamilton cycle:
every edge in the Hamilton cycle must go from red to blue or back, so at some point one colour has no points left and you cannot continue, while the other side still has points left or your cycle is not yet closed.

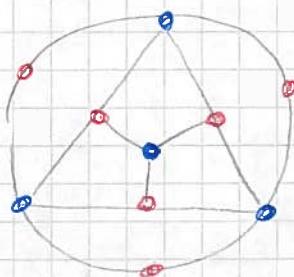
Example:



How can we see if this graph is bipartite?

Just start colouring one point, then give all its neighbours the other colour. Then give all their neighbours the other colour. If you ever encounter a point with the "wrong" colour, your graph is not bipartite.

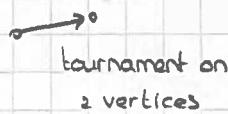
✓ General criterion: A graph G is bipartite if and only if it has no cycles of odd length



The graph in this example turns out to be bipartite, with 6 red points and 4 blue ones. Therefore, it has no Hamilton cycle.

Definition:

A tournament on n vertices, is a complete directed graph on n vertices. So for each $x, y \in V$ with $x \neq y$, exactly 1 of the edges (x, y) , (y, x) is in the graph.



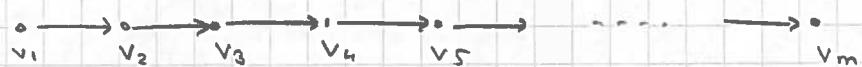
some possible tournaments on 4 vertices

Theorem 7: Any tournament has a ^{directed} Hamilton path

Proof:

We show that we can always create longer and longer paths until we reach a Hamilton path.

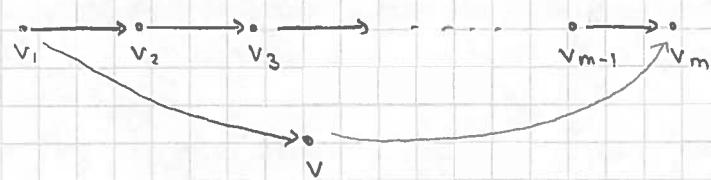
Suppose that we start with a path $p_m : v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \dots \rightarrow v_m$ of $m-1$ edges, where $m \geq 2$. (This is always possible for $m=2$: start with any edge). If $m=n$, we're done. If not, let v be a vertex that is not part of p_m .



If $(v, v_i) \in E$: we can make p_m longer.

If $(v_m, v) \in E$: we can make p_m longer.

Otherwise: we know $(v_1, v) \in E$ and $(v, v_m) \in E$:



If $(v, v_2) \in E$: create $v_1 \rightarrow v \rightarrow v_2 \rightarrow v_3 \rightarrow \dots \rightarrow v_m$ for a longer path
Otherwise: $(v_2, v) \in E$

Now if $(v, v_3) \in E$: create $v_1 \rightarrow v_2 \rightarrow v \rightarrow v_3 \rightarrow v_4 \rightarrow \dots \rightarrow v_m$ for a longer path
Otherwise: $(v_3, v) \in E$

And so on. Finally, the last "otherwise" says $(v_{m-1}, v) \in E$.
And then $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \dots \rightarrow v_{m-1} \rightarrow v \rightarrow v_m$ will give a longer path.
Done!

Example:

If we have a sports tournament with n players, where everyone plays exactly once against everyone else (and no matches end in a tie), there does not have to be a clear winner.

But we can list all the players in such a way that the first one beat the second one, the second one beat the third one, and so on. There could be multiple options for such a list!

Theorem d:
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Let $G = (V, E)$ be a loop-free graph with $|V| = n \geq 2$.

If $\deg(x) + \deg(y) \geq n-1$ for all $x, y \in V$ with $x \neq y$, then G has a Hamilton path

Proof:

See book.

For discussion in class:

- G connected (degree counting)
- Path lengthening at start or end
- Creation of a cycle

Corollary:

Let $G = (V, E)$ be a loop-free graph with $|V| = n \geq 2$. If $\deg(v) \geq \frac{n-1}{2}$ for all $v \in V$, then G has a Hamilton path

Proof:

If $x, y \in V$ with $x \neq y$, then $\deg(x) + \deg(y) \geq \frac{n-1}{2} + \frac{n-1}{2} = n-1$, so Theorem d holds.

Theorem g:
page 613

Let $G = (V, E)$ be a loop-free undirected graph with $|V| = n \geq 3$.
If $\deg(x) + \deg(y) \geq n$ for all non-adjacent and distinct $x, y \in V$, then G has a Hamilton cycle.

Proof:

See book

Corollary:

If $G = (V, E)$ is a loop-free undirected graph with $|V| = n \geq 3$ and $\deg(v) \geq n/2$ for all $v \in V$, then G has a Hamilton cycle

Proof:

Theorem g will hold in this case, since for all $x, y \in V$ we have $\deg(x) + \deg(y) \geq \frac{n}{2} + \frac{n}{2} = n$

Corollary:

If $G = (V, E)$ is a loop-free undirected graph with $|V| = n \geq 3$ and $|E| \geq \binom{n-1}{2} + 2$ then G has a Hamilton cycle

Proof: