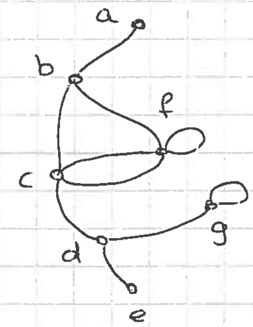
Lecture Plan Wednesday March 16: Graphs

Definition: page 582
Let $G = (V, E)$ be an undirected graph or multigraph.
For $v \in V$, the degree of v (written $\deg(v)$) is the number of edges that are incident with v, where a loop $\{v, v\}$ is counted twice.

Example.



$\deg(a) = 1$
$\deg(b) = 3$
$\deg(c) = 4$
$\deg(d) = 3$
$\deg(e) = 1$
$\deg(f) = 5$
$\deg(g) = 3$

Definition A vertex with degree 1 is called a pendant vertex.

Remark: Notice that $\deg(v)$ actually counts the number of "edge ends" that meet at v. Every edge has exactly 2 ends. This explains the theorem below:

Theorem 2: page 582
If $G = (V, E)$ is an undirected graph or multigraph: $\sum_{v \in V} \deg(v) = 2|E|$

Proof: Every edge has 2 "end pieces", so $2|E|$ counts the number of these endpieces.
$\deg(v)$ counts the number of endpieces that meet at v,
so $\sum_{v \in V} \deg(v)$ counts exactly every endpiece.

Hence $\sum_{v \in V} \deg(v) = 2|E|$.

Corollary. If $G = (V, E)$ is an undirected graph or multigraph, the number of vertices of odd degree must be even.

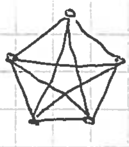
Proof: Suppose that G had k vertices of odd degree, where k was odd.
Then $\sum_{v \in V} \deg(v)$ is a sum of k odd numbers and some even numbers.

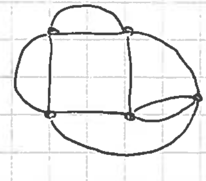
As k was odd, this sum must be odd. But $\sum_{v \in V} \deg(v) = 2|E|$, which is even! Contradiction. Therefore, G must have an even number of vertices of odd degree.

Definition: A graph or multigraph is called k-regular if all its vertices have degree k.

Example: Suppose we want a 4-regular graph with 10 edges.
Then $20 = 2 \cdot 10 = 2 \cdot |E| = \sum_{v \in V} \deg(v) = \sum_{v \in V} 4 = 4 \cdot |V|$
so $|V| = 5$.
In a graph, a 4-regular graph with 5 vertices can only be the complete graph K_5. In a multigraph, there are also other options.

K_5





one of the possible multigraphs with 10 edges that is 4-regular.

Example: One computer-related graph is the hypercube. Its vertices correspond to binary numbers (so 2^n vertices) and its edges correspond to binary numbers that only differ in one digit.

Q_0 :

(binary numbers with 0 digits)

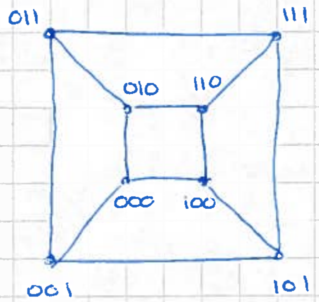
Q_1 :

(binary numbers with 1 digit)

Q_2 :

(binary numbers with 2 digits)

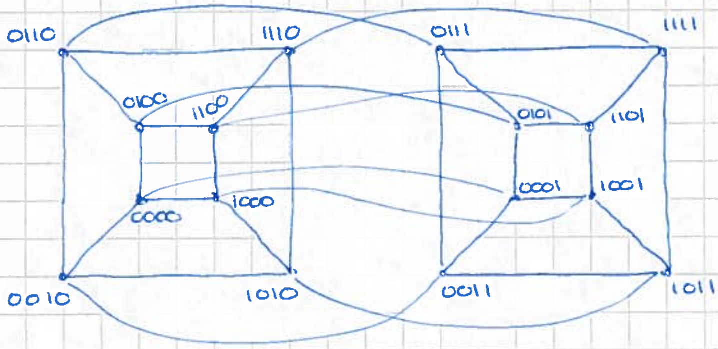
Q_3 :



(binary numbers with 3 digits)

this one could be seen as a cube in 3 dimensions!

Q_4



(binary numbers with 4 digits)

Hypercubes strike a nice balance between the number of edges and the length of shortest paths between points.

To go from vertex x to vertex y in Q_n, you need to change at most n digits, so you can go from x to y by a path of length at most n.

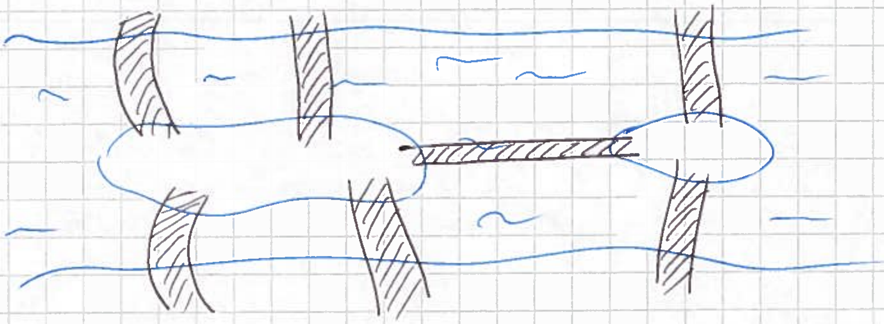
How many edges does Q_n have?

It has 2^n vertices and it is n-regular. By Theorem 2 we find:

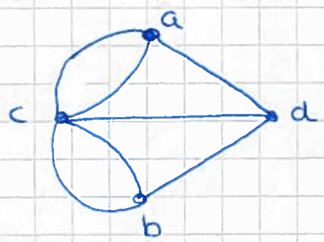
$$2|E| = \sum_{v\in V} \deg(v) = \sum_{v\in V} n = n\cdot|V| = n\cdot 2^n$$

So $|E| = \frac{n\cdot 2^n}{2} = n\cdot 2^{n-1}$.

Example: The seven bridges of Königsberg



Citizens of Königsberg wanted to walk around, go over each bridge exactly once, and end at their starting point.
Was this even possible?
Euler showed that it was not.



multi
Graph representation of the situation

Vertex a has odd degree. If we want to use these 3 bridges exactly once, we can

- leave a 3 times & never arrive (impossible)
- arrive once, leave twice (possible if we start at a, but then we cannot end there)
- arrive twice, leave once (possible if we end at a, but we cannot start there)
- leave 0 times and arrive 3 times (impossible)

See the theorem below for the general situation:

Definition: Let $G=(V,E)$ be an undirected graph or multigraph with no isolated vertices.
An Euler circuit in G is a circuit that uses every edge in G exactly once.
An Euler trail in G is a trail that uses every edge exactly once.

Theorem 3: Let $G=(V,E)$ be an undirected graph or multigraph with no isolated vertices.
page 586
Then G has an Euler circuit if and only if G is connected and every vertex in G has even degree.

Proof: Suppose first that G has an Euler circuit.
Since G has no isolated vertices, every vertex has an incident edge. This edge must be part of the Euler circuit. Therefore, every vertex is part of the Euler circuit. If we want to find a path from some vertex a to some vertex b, we can therefore start at a and follow the circuit until it reaches b. This shows that G is connected.
Suppose that our circuit starts and ends at vertex s. Then
- the circuit leaves x
- it may come back & leave again same number of times
- and finally it comes back to x.

So in the end, we leave just as many times as we come back, using different "edge ends" all the time. So $\deg(s)$ must be even.
For any other vertex x:
- as soon as we arrive, we leave again. This may happen multiple times

So once again, we leave & arrive the same number of times, so $\deg(x)$ must be even.
So all vertices in G have even degree.

Now assume that G is connected and all its vertices have even degree.
We prove that G has an Euler circuit by induction on $|E|$.

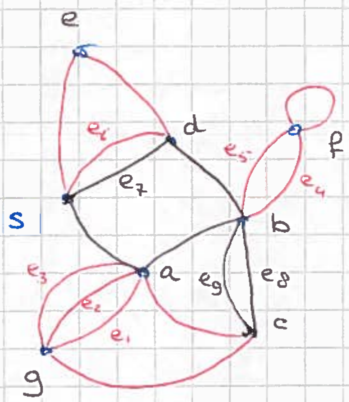
Basic step: If $|E| = 1$, then G looks like: 
so there is an Euler circuit.

If $|E| = 2$, then G looks like  or 
so there is an Euler circuit.

Now let $n \in \mathbb{N}$ with $n \geq 2$ and assume as an induction hypothesis that all connected graphs with n edges whose vertices all have even degrees, have Euler circuits.

Induction step: Suppose that G is a connected graph with $n+1$ edges, whose vertices all have even degrees.
We must show that G has an Euler circuit.

First choose a vertex $s \in V$ as a starting point. Create any circuit from s to s (since all vertices of G have even degrees, you cannot get stuck at a vertex: if you arrive, you can leave by a new edge).
Call this circuit C.
Remove the edges of C from G and remove all vertices that have become isolated. Call the new graph G'.
G' may have multiple connected components. The vertices in G' still have even degrees.
Since the connected components of G' have $\leq n$ edges, they must have euler circuits.
And all these euler circuits are connected, somewhere, to the circuit C since the original graph was connected.
Hence we can combine the circuits to form one big Euler circuit.



$C:\ s \xrightarrow{e_7} d \to b \xrightarrow{e_8} c \xrightarrow{e_9} b \to a \to c$

New circuit: We see an Euler circuit from a component of G' at s, so follow that first.

$s \to e \to d \xrightarrow{e_6} s \xrightarrow{e_7} d \to b \xrightarrow{e_4} f \to f \xrightarrow{e_5} b \xrightarrow{e_8} c \to a \xrightarrow{e_1} g \xrightarrow{e_2} a \xrightarrow{e_3} g \to c \xrightarrow{e_9} b \to a \to s$

Corollary: If G is an undirected graph or multigraph with no isolated vertices, then G has an open Euler trail if and only if G is connected and has exactly 2 vertices of odd degree.

Proof: Suppose first that G is connected and that exactly 2 vertices a and b have odd degree. Add an extra edge $\{a,b\}$ to G and call this G'. Now all vertices in G' have even degree, so G' has an Euler circuit. If we remove our new edge $\{a,b\}$ from this circuit, it becomes an Euler trail from a to b.
Conversely, if G has an open Euler trail, let a and b be its endpoints. Add an extra edge $\{a,b\}$ to G and call this new graph G'. If we add the new edge $\{a,b\}$ to the Euler trail from a to b, it becomes an euler circuit in G'. Hence all vertices in G' have even degree. Therefore, a and b must have had odd degrees in G and all the other vertices must have had even degrees.
The Euler trail in G shows that G is connected since it connects all vertices to each other (G had no isolated vertices).

Remark: Since the Königsberg problem had a graph with 4 vertices of odd degree, it has no Euler circuit and no Euler trail.

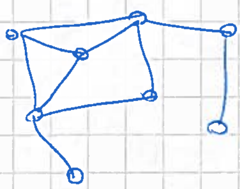
Definition: Let $G=(V,E)$ be a directed graph or multigraph, and let $v \in V$.
The incoming degree (or indegree) of v is the number of edges incident to v. It is denoted by $id(v)$
The outgoing degree (or outdegree) of v is the number of edges incident from v. It is denoted by $od(v)$.

Theorem 4: page 587
Let $G=(V,E)$ be a directed graph or multigraph with no isolated vertices. Then G has a directed Euler circuit if and only if G is connected and $id(v)=od(v)\ \forall v \in V$.

Proof: skipped.

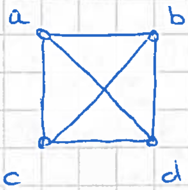
Definition: A graph is called planar if it can be drawn such that its edges only intersect each other at vertices.
Such a drawing is called an embedding of G in the plane.

Example: G_1



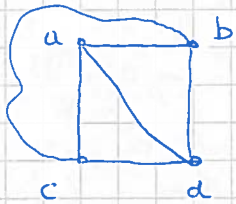
A planar graph embedded in the plane

G_2



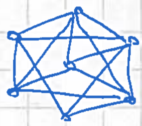
Not an embedding, but this does not mean that the graph is not planar

G_2

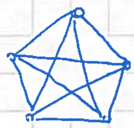


This shows that the graph G_2 is planar, since we have found an embedding in the plane.

Example: Do you think this graph could be planar?

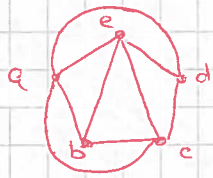


Or this one:



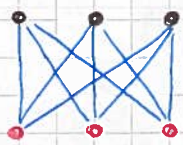
K_5

Try to improve the drawing



I get stuck on the last edge, that should go from a to c

And what about this one?

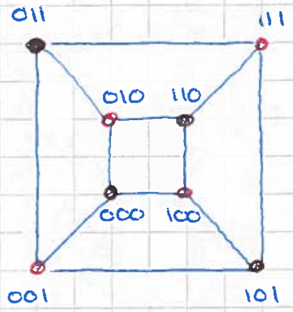


We have edges from all the black points to all the red points.

Definition: A graph $G=(V,E)$ is called bipartite if we can write $V=V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ and every edge of G is of the form $\{a,b\}$ with $a \in V_1$ and $b \in V_2$.
(Idea: G is bipartite if we can colour its points red and black in such a way that every edge is between a red and a black point)

If $|V_1| = m$ and $|V_2| = n$ and we have an edge from every point in V_1 to every point in V_2, this is called a complete bipartite graph and it is denoted by $K_{m,n}$.

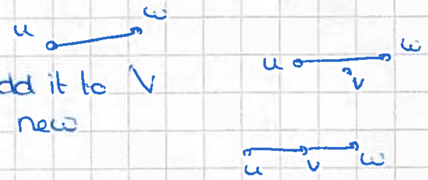
Example: The hypercube Q_n is bipartite: colour a vertex red if it has an even number of 1's, and black if it has an odd number of 1's.
Any   in Q_n corresponds to changing one digit in a vertex. This will always change an odd number of 1's to an even number, or the other way around. So any   runs between a red and a black point.



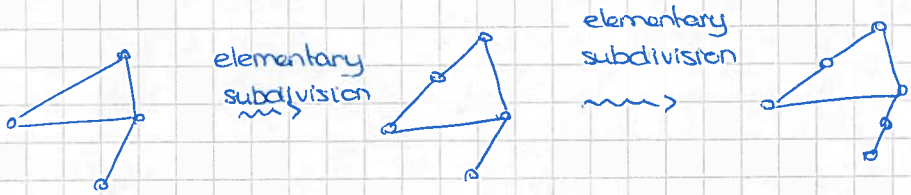
Remark: So far, we have seen 2 pretty simple graphs (K_5 and $K_{3,3}$) that we did not manage to find an embedding in the plane for.

Definition: Let $G=(V,E)$ be a loop-free undirected graph with $|E| \neq 0$.
An elementary subdivision of G is what you get when you

- pick an edge $\{u,w\} \in E$
- create a new point v and add it to V
- replace the edge $\{u,w\}$ by two new edges: $\{u,v\}$ and $\{v,w\}$

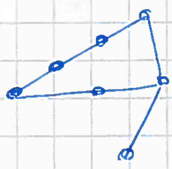


Example:



Definition: If $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are two loop-free undirected graphs, they are called homeomorphic if they are isomorphic or can be obtained from the same loop-free undirected graph H by a sequence of elementary subdivisions.

Example:



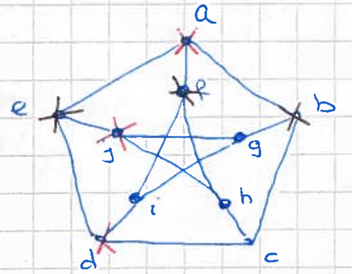
and

are homeomorphic

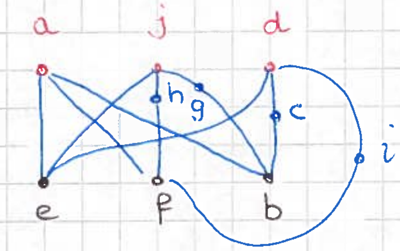
Theorem: A graph G is non-planar if and only if it contains a subgraph which is homeomorphic to either K_5 or $K_{3,3}$.

Proof: Not included in book and course.
The theorem was proved by Kuratowski

Example: Petersen graph

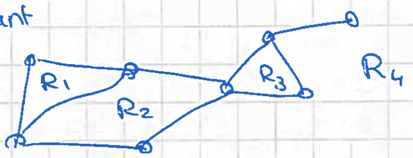


We find a graph homeomorphic to $K_{3,3}$ hidden inside



Remark: We have never actually proven that K_5 and $K_{3,3}$ are non-planar. That is what we will be working towards now.

Notation: A planar embedding of a graph cuts up the plane into several regions, where we count the "outside" as a region as well. (called "the infinite region")



We can only do this for planar embeddings!

Theorem 6: Let $G=(V,E)$ be a connected planar graph (or multigraph) with $|V|=v$ and $|E|=e$. Let r be the number of regions in the plane given by a planar embedding of G.
Then $v-e+r=2$

Proof: We use induction on e.

Basic step: If $e=0$, $G = \bullet a$ so $|V|=1$, $|E|=0$, $r=1$ and $v-e+r=2$.

If $e=1$, $G =$  so $|V|=1$, $|E|=1$, $r=2$ and $v-e+r=2$

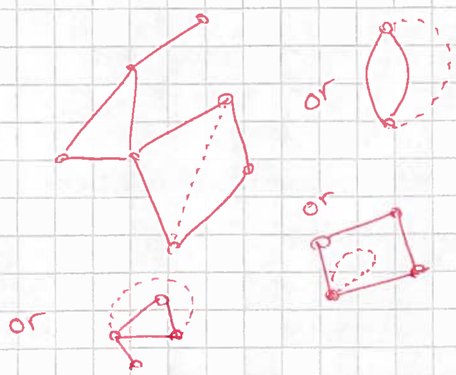
or

$G =$  so $|V|=2$, $|E|=1$, $r=1$ and $v-e+r=2$

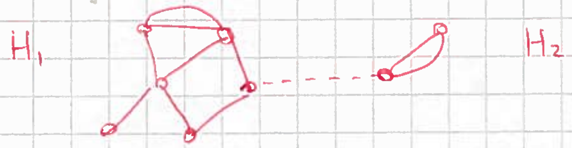
Now let $k\in\mathbb{N}$ and assume as an induction hypothesis that the result is true for all connected planar (multi)graphs with at most k edges.

Induction step: Let $G=(V,E)$ be a connected planar (multi)graph with $k+1$ edges (we need to prove that $v-e+r=2$).
Since G has $k+1$ edges, it has at least one edge $\{a,b\}$.
Let $H = G-\{a,b\}$ be the subgraph you get by deleting this edge.
H can be connected or disconnected.

- If H is connected, we know $v_H - e_H + r_H = 2$ by the induction hypothesis.
  The edge $\{a,b\}$ must go through some region and cut it in two parts.
  So $v_G = v_H$, $e_G = e_H+1$, $r_G = r_H+1$ and $v_G - e_G + r_G = v_H-(e_H+1)+(r_H+1)=2$.



- If H is not connected, it must have 2 components H_1 and H_2 that were connected to each other by edge $\{a,b\}$



By the induction hypothesis: $v_{H_1} - e_{H_1} + r_{H_1} = 2$

$v_{H_2} - e_{H_2} + r_{H_2} = 2$

Now notice that $v_G = v_{H_1} + v_{H_2}$, $e_G = e_{H_1} + e_{H_2} + 1$
and $r_G = \#\{\text{regions of } H_1 \text{ that are not infinite}\}$
$+ \#\{\text{regions of } H_2 \text{ that are not infinite}\}$
$+ \#\{\text{the infinite regio}\}$
$= (r_{H_1} - 1) + (r_{H_2} - 1) + 1 = r_{H_1} + r_{H_2} - 1$

So $v_G - e_G + r_G = (v_{H_1} + v_{H_2}) - (e_{H_1} + e_{H} + 1) + (r_{H_1} + r_{H_2} - 1)$

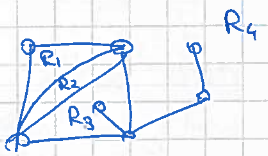
$= v_{H_1} + v_{H_2} - e_{H_1} - e_{H_2} - 1 + r_{H_1} + r_{H_2} - 1$

$= (v_{H_1} - e_{H_1} + r_{H_1}) + (v_{H_2} - e_{H_2} + r_{H_2}) - 2$

$= 2 + 2 - 2 = 2.$

This finishes the induction and proves the theorem.

Definition: If R is a region in a planar embedding of a graph or multigraph, the degree of R is the number of edges in a shortest closed walk along the edges of the boundary of R. It has to include all the edges in this boundary



$\deg(R_1) = 3$
$\deg(R_2) = 2$
$\deg(R_3) = 5$
$\deg(R_4) = 8$

Corollary: Let $G = (V, E)$ be a loop-free connected planar graph with $|V| = v$, $|E| = e$ and r regions, where $e > 2$.
Then $3r \leq 2e$ and $e \leq 3v - 6$

Proof: G is loop-free and not a multigraph, and has at least 2 edges.
The boundary of any region (including the infinite region) has at least 3 edges in this setting. So every region has degree ≥ 3.
$2e = 2|E| =$ the sum of the degrees of all the regions $\geq 3r$
so $3r \leq 2e$.
And by Euler's theorem: $2 = v - e + r \leq v - e + (\frac{2}{3})e = v - (\frac{1}{3})e$
so $6 \leq 3v - e$ and $e \leq 3v - 6$.

Remark: If $G=(V,E)$ is a loop-free and connected graph with $|E| = e > 2$:

$$G \text{ planar} \implies 2e \geq 3r \ \& \ e \leq 3v-6$$
$$\not\Longleftarrow$$

$$2e < 3r \text{ or } e > 3v-6 \implies G \text{ not planar}$$
$$\not\Longleftarrow$$

Remark: K_5 is loop-free and connected. It is a graph with $|E| = 10 > 2$.

$v = |V| = 5$
$e = |E| = 10$
$\Big\}$ $3v-6 = 15-6 = 9 < 10 = e$, so K_5 is not planar

$K_{3,3}$ is a loop-free connected graph with $|E| = 9 > 2$.

$v = |V| = 6$
$e = |E| = 9$
So $3v-6 = 18-6 = 12 > 9 = e$. We can conclude nothing here.
But as $K_{3,3}$ is a bipartite graph, it cannot contain 3-cycles.
Therefore, any region in a planar embedding would have degree ≥ 4.
So $4r \leq 2e$ in this case.
From Euler's theorem: $v - e + r = 2$, so $r = e - v + 2 = 9 - 6 + 2 = 5$.
Then $4r = 20$ and $2e = 18$.
This proves that $K_{3,3}$ is not planar.