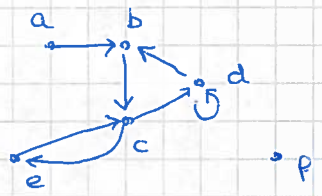
Lecture plan Monday March 14 2016: Graphs

Definition: (page 566)

A directed graph (or digraph) is a pair (V,E)
where V is a finite, non-empty set (the set of vertices of the digraph)
and $E \subseteq V \times V$ is a subset (the set of directed edges of the digraph)
If we call our directed graph G, we write $G=(V,E)$

An undirected graph is a pair (V,E)
where V is a finite, non-empty set
and E is a set of unordered pairs of elements from V.

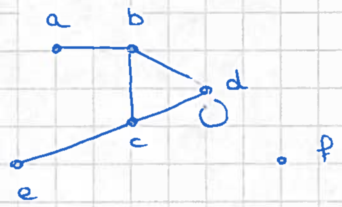
Example: Directed graph:



$V = \{a,b,c,d,e,f\}$

$E = \{(a,b),(b,c),(c,d),(d,b),(d,d),(e,c),(c,e)\}$

Undirected graph:



$V = \{a,b,c,d,e,f\}$

$E = \{\{a,b\},\{b,c\},\{c,d\},\{d,d\},\{d,b\},\{e,c\}\}$

Terminology: In both graphs above:

- Edge (b,c) is incident with vertices b and c
- Edge (d,d) (or $\{d,d\}$) is called a loop
- Vertex f is isolated, since it has no incident edges

In the directed graph:

- Vertex b is adjacent to vertex c
- Vertex c is adjacent from vertex b
- Vertex b is the origin (or source) of edge (b,c)
- Vertex c is the terminus (or terminating vertex) of edge (b,c)

A graph without loops is called loop-free.
If it is not specified if a graph is directed or not, we assume it to be undirected.

Remark: If $G=(V,E)$ is a directed graph, we can turn it into an undirected graph (called "the undirected graph associated to G") by removing all the directions from the edges.

If E contains both (a,b) and (b,a), then the associated undirected graph contains just one unordered pair $\{a,b\}$.
See the example above for an example.

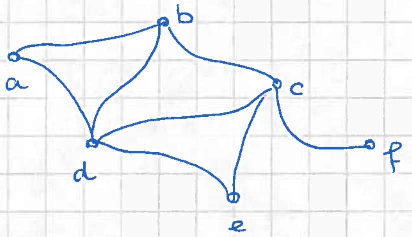
Definition: Let $G=(V,E)$ be an undirected graph, and let $x,y \in V$ be two (not necessarily distinct) points.

An x-y walk in G is a (loop-free) alternating sequence

$$x = x_0,\ e_1,\ x_1,\ e_2,\ x_2,\ \dots\ ,\ x_{n-1},\ e_n,\ x_n = y$$

where $x_0, \dots, x_n \in V$ and $e_i = \{x_{i-1}, x_i\} \in E$ for $i = 1, \dots, n$.
The number n is called the length of the walk.
When $n=0$, the walk is called trivial.
When $n \geq 1$ and $x=y$, the walk is called a closed walk.
When $n \geq 1$ and $x \neq y$, the walk is called an open walk.

Example:



G

$a, \{a,b\}, b, \{b,d\}, d, \{d,c\}, c, \{c,e\}, e, \{e,d\}, d, \{d,b\},$ is an a-b walk.
We can write it down shorter as:
$\{a,b\}, \{b,d\}, \{d,c\}, \{c,e\}, \{e,d\}, \{d,b\}$
or as
$a \to b \to d \to c \to e \to d \to b$
The walk is open and has length 6.
Notice that this walk repeats some edges ($\{b,d\}$) and some vertices (b & d).

Since G is undirected, we consider our walk to be an b-a walk as well, by reading it backwards.

Definition: If an x-y walk repeats no edges, it is called a trail.
If a trail is closed, it is called a circuit.

If an x-y walk repeats no vertices (except possibly at the start & end when $x=y$), it is called a path.
A closed path is called a cycle.

Example: Let students suggest some trails, circuits, paths and cycles in the graph G above.

Remark: In a directed graph, we have directed walks/trails/circuits/paths/cycles that consist of directed edges (used in the right direction)

Summary table:

Name	Open	Closed	Repeated edges allowed	Repeated vertices allowed
Open walk	✓		✓	✓
Closed walk		✓	✓	✓
Open trail	✓		—	✓
Circuit = closed trail		✓	—	✓
Open path	✓		—	—
Cycle = closed path		✓	—	—

Theorem 1: page 569

Let $G=(V,E)$ be an undirected graph, with $a,b \in V$ and $a \neq b$. If there is a trail in G from a to b, then there must also be a path in G from a to b.

Proof:

Since there are one or multiple trails from a to b, we can select one of the shortest possible length.
Write it as $a \to x_1 \to x_2 \to \dots \to x_n \to b$. Write $a = x_0$ and $b = x_{n+1}$.
If this trail is not a path, then $x_k = x_m$ for some k and m with $0 \leq k < m \leq n+1$.
But then we can create a shorter trail from a to b:

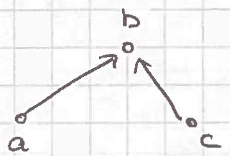
$$x_0 \to x_1 \to \dots \to x_k = x_m \to x_{m+1} \to \dots \to x_{n+1}$$

Contradiction! So the original shortest trail must have been a path.

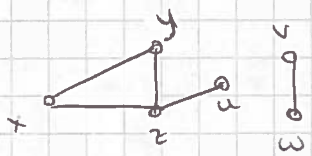
Definition:

Let $G=(V,E)$ be a graph. We call G connected if there is a path between any two distinct vertices of G. If G is not connected, we all it disconnected.
A digraph is called connected if its associated undirected graph is connected.

Example:



This digraph is connected, even though ther is no directed path from a to c.
After all, its associated undirected graph is connected.



This graph is not connected, because there is no path from u to v.
However, we can split it up into two pieces that are connected:

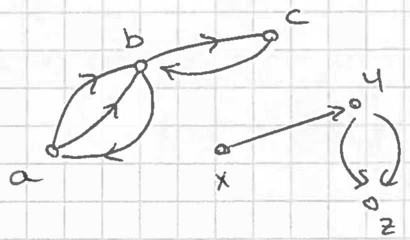
$V_1 = \{x, y, z, u\}$
$E_1 = \{\{x,y\}, \{y,z\}, \{x,z\}, \{z,u\}\}$
$V_2 = \{v, w\}$
$E_2 = \{\{v,w\}\}$

These two pieces are called the connected components of the graph.

Definition: If $G = (V, E)$ is a graph, we denote its number of connected components by $\kappa(G)$

Definition: A multigraph is a graph where edges may occur multiple times. There are directed and undirected multigraphs.
The number of times that an edge occurs in a multigraph (directed or undirected) is called the multiplicity of the edge.

Example:

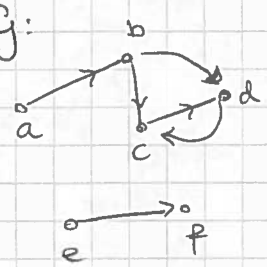


A directed multigraph with 2 connected components. Edge (a,b) has multiplicity 2, while edge (b,a) has multiplicity 1.

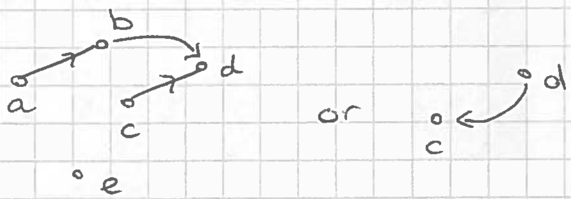
Definition: If $G = (V, E)$ is a graph (either directed or undirected), then $G_1 = (V_1, E_1)$ is called a subgraph of G if
- $V_1 \subseteq V$
- $E_1 \subseteq E$
- The edges in E_1 are incident with vertices in V_1.

Example: A graph and some subgraphs

G:



Some subgraphs of G:



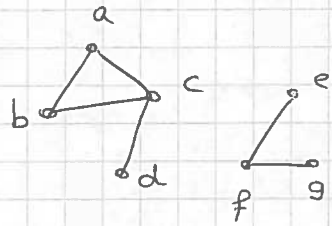
Terminology: If $G = (V, E)$ is a (directed or undirected) graph and $G_1 = (V_1, E_1)$ is a subgraph, then G_1 is called a spanning subgraph if $V_1 = V$

So basically, a spanning subgraph of G is obtained by keeping all vertices and removing some (or none, or all) of the edges of G.

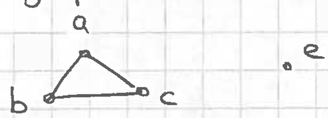
Definition: Let $G = (V, E)$ be a (directed or undirected) graph.
If $\emptyset \neq U \subseteq V$ is a subset of V, then the subgraph of G induced by U (notation: $\langle U \rangle$) is the graph with
- all the vertices in U
- all the edges in G that are between vertices of U.

If G_1 is a subgraph of G, we call it an induced subgraph if $G_1 = \langle U_1 \rangle$ for some subset $U_1 \subseteq V$.

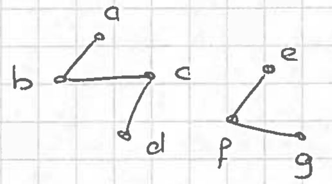
Example: G :



Some subgraphs:



This is not a spanning subgraph.
It is an induced subgraph, induced by $\{a,b,c,e\}$

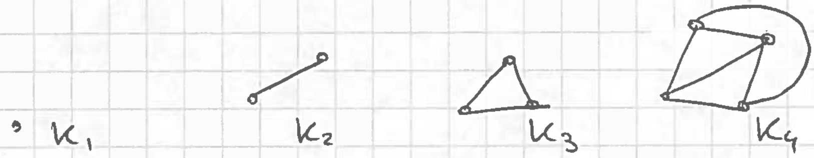


This is a spanning subgraph.
It is not an induced subgraph, since it has the vertices a and c but not edge $\{a,c\}$.

Definition: Let $G=(V,E)$ be a graph (directed or undirected)

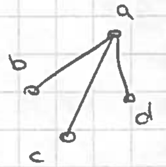
- If $v \in V$, then $G-v$ denotes the subgraph induced by $V-\{v\}$. This is the graph that you get when you remove v and all the edges that are incident with v
- If $e \in E$, then $G-e$ denotes the subgraph that you get when you remove edge e from G. (You keep all vertices).
- (We can also construct subgraphs like $G-\{e_1,e_2,e_3\}$ if we remove more than one edge.

Definition: Let V be a set of n vertices. The complete graph on V (denoted K_n) is the loop-free, undirected graph where all points $a,b \in V$ with $a \neq b$ are connected by an edge $\{a,b\}$.

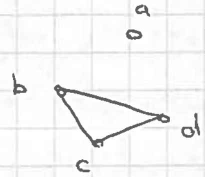


Definition: If G is a loop-free undirected graph on n vertices, we can define its complement $\overline{G}$. This is the subgraph of K_n with all the vertices of G and all the edges that are not in G.

Example: G :



$\overline{G}$



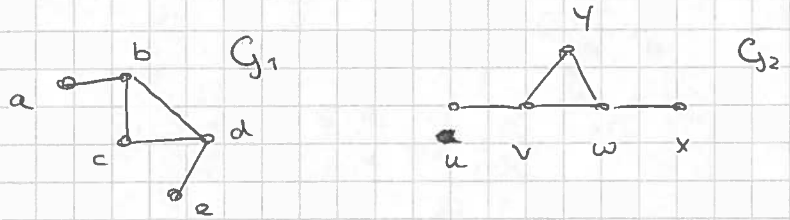
Remark: A graph with 0 edges is called a null graph. Its complement will be a complete graph.

Definition: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs.
A function $f: V_1 \to V_2$ is called a graph isomorphism if

- f is bijective (so f is one-to-one and onto)
- $\forall a,b \in V_1 : \{a,b\} \in E_1 \iff \{f(a), f(b)\} \in E_2$

When such a function f exists, we call G_1 and G_2 isomorphic to each other.

Example:



$f: V_1 \to V_2$ given by $f(a)=x$, $f(b)=y$, $f(c)=u$, $f(d)=v$, $f(e)=w$
is not an isomorphism: $\{a,b\} \in E_1$ but $\{f(a), f(b)\} = \{x,y\} \notin E_2$.

$g: V_1 \to V_2$ given by $g(a)=u$, $g(b)=v$, $g(c)=y$, $g(d)=w$, $g(e)=x$
is an isomorphism:

E_1		E_2
$\{a,b\}$	$\longleftrightarrow$	$\{g(a), g(b)\} = \{u,v\}$
$\{b,c\}$	$\longleftrightarrow$	$\{g(b), g(c)\} = \{v,y\}$
$\{b,d\}$	$\longleftrightarrow$	$\{g(b), g(d)\} = \{v,w\}$
$\{c,d\}$	$\longleftrightarrow$	$\{g(c), g(d)\} = \{y,w\}$
$\{d,e\}$	$\longleftrightarrow$	$\{g(d), g(e)\} = \{w,x\}$

Remark: "To find an isomorphism between the graphs, you must reconfigure them to make them match each other".
In the example above, you can see that you have to map the triangle to the triangle (otherwise the edges do not match)

More general: if $f: V_1 \to V_2$ is an isomorphism of two graphs G_1 and G_2, and G_1 contains a cycle of length k, then f mus map this cycle to a cycle of length k in G_2.
This concept can help you find isomorphisms

It can also help you show that two graphs are not isom

Not isomorphic since the first graph has a cycle of length 3 and the second one does not.

Not isomorphic since the number of edges does not match

Not isomorphic since the first one has a vertex with 3 incident edges and the second one does not.