Lecture plan mandag 22 februar 2016 : Functions

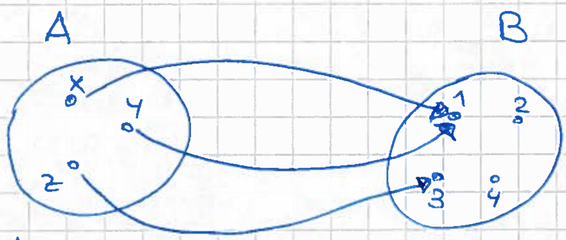
Definition: Let A and B be non-empty sets.
A function from A to B is a relation $f \subseteq A \times B$ such that for each $a \in A$ there is exactly one pair $(a, \text{something})$ in f with a as its first component.

Notation: We write $f(a) = b$ when (a,b) is the unique pair in f with a as its first component. We call b the "image of a under f" and a a "pre image" of b.

A is called the domain of f, and B its codomain.
The range of f is denoted by $f(A)$ and defined by

$$f(A) = \{ b \in B \mid \exists a \in A,\ b = f(a) \}$$
$$= \{ f(a) \mid a \in A \}$$

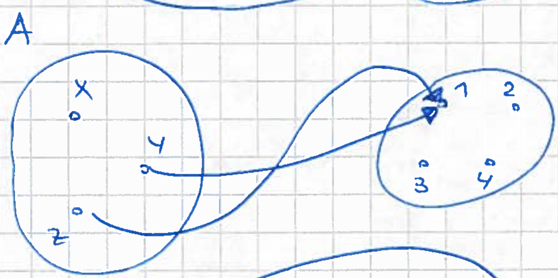
Example: a)



This is a function.
$f = \{ (x,1), (y,1), (z,3) \}$

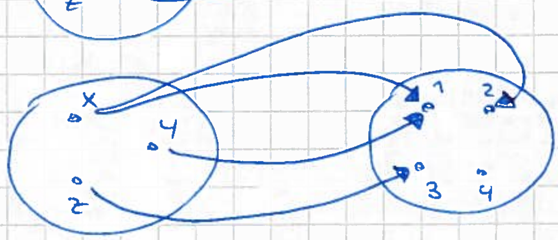
$f(x) = 1, \quad f(y) = 1, \quad f(z) = 3$

b)



Not a function, since x has no image.

c)



Not a function, since x has two distinct images

The function in (a) has range $\{1,3\}$. The others are not functions.

Example: 1) $f: \mathbb{Z} \to \mathbb{Z}, \qquad f(n) = n^2$

Other notation: $f = \{ (n, n^2) \mid n \in \mathbb{Z} \} \subseteq \mathbb{Z} \times \mathbb{Z}$

2) $f: \mathbb{Q} \to \mathbb{R}, \qquad f(n) = n^2$

This is not the same function as (1)!

Remark: $f: \mathbb{R} \to \mathbb{Q}, \quad f(n) = n^2$ is not a function!

$\sqrt[4]{2} \in \mathbb{R}$ has no image in $\mathbb{Q}$.

Examples:

3) The floor function $\lfloor\ \rfloor : \mathbb{R} \to \mathbb{Z}$ given by

$\lfloor x \rfloor =$ the greatest integer less than or equal to x.

$=$ the unique integer n for which $n \leq x < n+1$

$=$ the unique integer n for which $x-1 < n \leq x$

$\lfloor 3{,}72 \rfloor = 3$, $\lfloor -2{,}4 \rfloor = -3$, $\lfloor -2 \rfloor = -2$ etc.

Note the following:

$$\lfloor 7{,}1 + 8{,}2 \rfloor = \lfloor 15{,}3 \rfloor = 15 = 7+8 = \lfloor 7{,}1 \rfloor + \lfloor 8{,}2 \rfloor$$

but

$$\lfloor 7{,}7 + 8{,}4 \rfloor = \lfloor 16{,}1 \rfloor = 16 \neq 7+8 = \lfloor 7{,}7 \rfloor + \lfloor 8{,}4 \rfloor$$

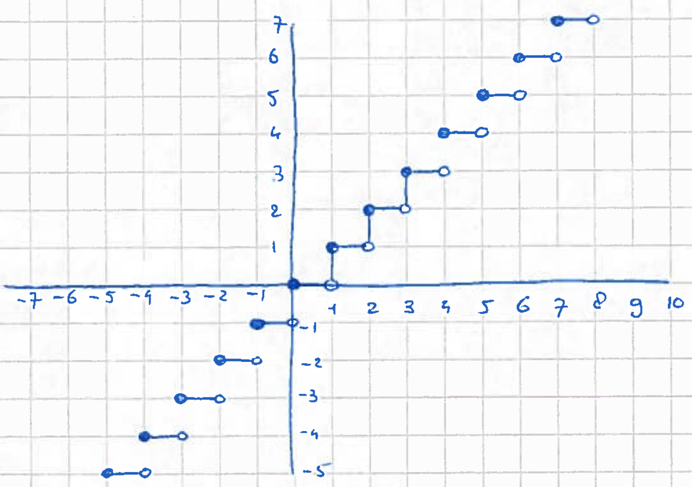
and also

$$\lfloor 3 \cdot 2{,}1 \rfloor = \lfloor 6{,}3 \rfloor = 6 = 3 \cdot 2 = 3 \cdot \lfloor 2{,}1 \rfloor$$

but

$$\lfloor 3 \cdot 2{,}4 \rfloor = \lfloor 7{,}2 \rfloor = 7 \neq 3 \cdot 2 = 3 \cdot \lfloor 2{,}4 \rfloor$$

We can draw a part of the graph:



Domain: $\mathbb{R}$
Codomain: $\mathbb{Z}$
Range: for any $x \in \mathbb{R}$, $\lfloor x \rfloor$ is an integer so $\lfloor x \rfloor \in \mathbb{Z}$.
This proves that the range is a subset of $\mathbb{Z}$.
For any $n \in \mathbb{Z}$, we have $n \in \mathbb{R}$ and $\lfloor n \rfloor = n$,
so n is an element of the range. This proves
that $\mathbb{Z}$ is a subset of the range.
So the range is $\mathbb{Z}$.

4) The ceiling function $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ given by

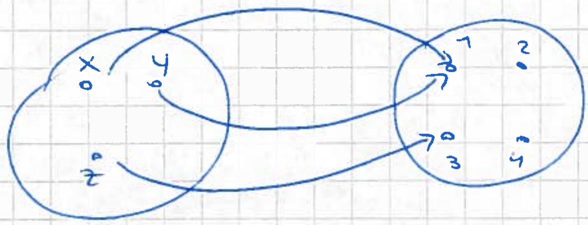
$\lceil x \rceil =$ the smallest integer that is greater than or equal to x

$=$ the unique integer n for which $n-1 < x \leq n$

$=$ the unique integer n for which $x \leq n < x+1$

5) A sequence $a_0, a_1, a_2, a_3, \ldots$ with $a_i \in \mathbb{R}$ for $i = 0,1,2,\ldots$ can be regarded as a function $f : \mathbb{N} \to \mathbb{R}$ with $f(n) = a_n$.

Definition: A function $f : A \to B$ is called injective (or "one-to-one") if all elements of A have different images



Not injective, since x and y have the same image.

Example: $f : \mathbb{Z} \to \mathbb{Z}$ with $f(n) = n^2$ is not injective, since $f(-1) = 1 = f(1)$.

$g : \mathbb{N} \to \mathbb{Z}$ with $g(n) = n^2$ is injective:
suppose $n, m \in \mathbb{N}$ with $n \neq m$. Then $m < n$ or $n < m$.

- If $m < n$, then $m^2 < n^2$ since m and n are ≥ 0. Hence $m^2 \neq n^2$.
- If $n < m$, then $n^2 < m^2$ since m and n are ≥ 0. So $m^2 \neq n^2$.

In both cases, we find $m^2 \neq n^2$, so $g(m) \neq g(n)$.

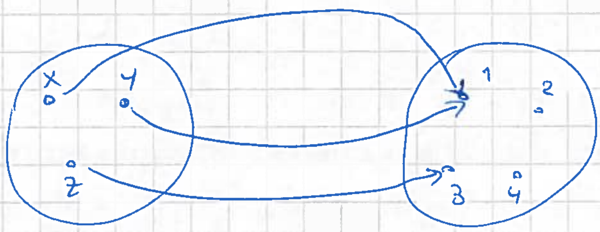
The floor and ceiling functions are not injective, since $\lfloor 3{,}2 \rfloor = \lfloor 3{,}3 \rfloor$ and $\lceil 3{,}2 \rceil = \lceil 3{,}3 \rceil$.

$h : \mathbb{R} \to \mathbb{R}$ with $h(x) = 3x - 2$ is injective:
suppose $x, y \in \mathbb{R}$. If $h(x) = h(y)$, then $3x - 2 = 3y - 2$. This implies $3x = 3y$, and hence $x = y$. (We proved the contrapositive: "if $h(x) = h(y)$, then $x = y$" which is equivalent to the statement "if $x \neq y$, then $h(x) \neq h(y)$".)

Definition: If $f : A \to B$ is a function and $X \subseteq A$ is a subset, then the image of X under f is

$$f(X) = \{ b \in B \mid \exists x \in X,\ b = f(x) \}$$

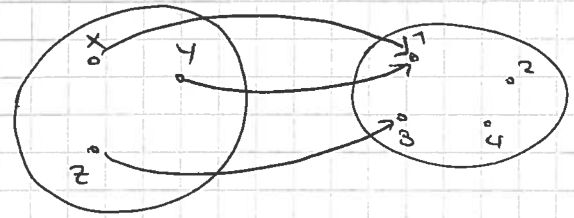
$$= \{ f(x) \mid x \in X \}$$



$f(\{x,y\}) = \{1\}$
$f(\{x,z\}) = \{1,3\}$
$f(\{x\}) = \{1\}$
$f(x) = 1$

The image of a set is a set, the image of an element is an element.

Definition: A function $f: A \to B$ is called surjective (or "onto") if for every $b \in B$ there exists $a \in A$ with $f(a) = b$.
So f is onto if and only if $f(A) = B$.



Not surjective since 2 and 4 are not in the range of f.

Examples:

- $f: \mathbb{Z} \to \mathbb{Z}$ with $f(n) = n^2$ is not onto, since $-1 \notin f(\mathbb{Z})$
- $f: \mathbb{R} \to \mathbb{R}$ with $f(r) = 3r+4$ is onto:
  Let $b \in \mathbb{R}$. I want to find $r \in \mathbb{R}$ such that $3r+4 = b$.
  This is possible by taking $r = \dfrac{b-4}{3}$. Then $r \in \mathbb{R}$ and $3r+4 = b$
  so $b = f(r) \in f(\mathbb{R})$.
  This proves that $\mathbb{R} \subseteq f(\mathbb{R})$. Since we know that $f(\mathbb{R}) \subseteq \mathbb{R}$, we now have $f(\mathbb{R}) = \mathbb{R}$ so f is surjective.
- $f: \mathbb{Z} \to \mathbb{Z}$ with $f(n) = 3n+4$ is not surjective:
  For $n \leq 0$ we have $f(n) = 3n+4 \leq 4$.
  For $n > 0$ we have $n \geq 1$ and $f(n) = 3n+4 \geq 7$.
  So $6 \notin f(\mathbb{Z})$ and hence f is not onto.

Definition: (page 283) If $f: A \to B$ is a function and $X \subseteq A$ is a subset, then the restriction of f to X is defined by

$$f|_X : X \to B \qquad f|_X(a) = f(a) \quad \forall a \in X$$

Definition: (page 283) If $X \subseteq A$ and $f: X \to B$ and $g: A \to B$ are functions, then g is called an extension of f to A if $g(a) = f(a)$ $\forall a \in X$.

Example:
$A = \{1, 2, 3, 4, 5\}$
$f: A \to \mathbb{R}$ $\quad f = \{(1,10), (2,13), (3,16), (4,19), (5,22)\}$
$g: \mathbb{Q} \to \mathbb{R}$ $\quad g(q) = 3q+7$ for all $q \in \mathbb{Q}$.
$h: \mathbb{R} \to \mathbb{R}$ $\quad h(r) = 3r+7$ for all $r \in \mathbb{R}$

g is an extension of f to $\mathbb{Q}$
h is an extension of g to $\mathbb{R}$
h is an extension of f to $\mathbb{R}$
g is the restriction of h to $\mathbb{Q}$
f is the restriction of g to A
f is the restriction of h to A.

Theorem 2: Let $f: A \to B$ be a function, and let $X \subseteq A$ and $Y \subseteq A$ be subsets of A.
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a) $f(X \cup Y) = f(X) \cup f(Y)$

b) $f(X \cap Y) \subseteq f(X) \cap f(Y)$

c) $f(X \cap Y) = f(X) \cap f(Y)$ if f is injective.

Proof:

a) Suppose $b \in f(X \cup Y) = \{ f(a) \mid a \in X \cup Y \}$. Then there is an element $u \in X \cup Y$ with $f(u) = b$. Since $u \in X \cup Y$, we have $u \in X$ or $u \in Y$.
- If $u \in X$, then $b = f(u) \in f(X) \subseteq f(X) \cup f(Y)$
- If $u \in Y$, then $b = f(u) \in f(Y) \subseteq f(X) \cup f(Y)$

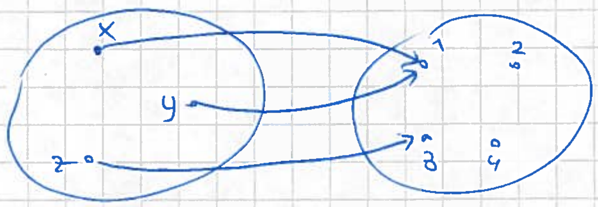
So in any case, we find $b \in f(X) \cup f(Y)$. This proves that $f(X \cup Y) \subseteq f(X) \cup f(Y)$.
Now suppose $c \in f(X) \cup f(Y)$. Then $c \in f(X)$ or $c \in f(Y)$ (or both).
- If $c \in f(X)$, then $c = f(x)$ for some $x \in X$.
  Now $x \in X \cup Y$ and hence $c = f(x) \in f(X \cup Y)$
- If $c \in f(Y)$, then $c = f(y)$ for some $y \in Y$.
  Now $y \in X \cup Y$ and hence $c = f(y) \in f(X \cup Y)$

So in any case, we find $c \in f(X \cup Y)$. This proves that $f(X) \cup f(Y) \subseteq f(X \cup Y)$.
Together, these two inclusions prove (a).

b) Suppose $b \in f(X \cap Y)$. Then there exists $a \in X \cap Y$ with $b = f(a)$.
Since $a \in X$, we have $b = f(a) \in f(X)$.
Since $a \in Y$, we have $b = f(a) \in f(Y)$.
So $b \in f(X) \cap f(Y)$. This proves that $f(X \cap Y) \subseteq f(X) \cap f(Y)$

c) We already know (b), and we now assume that f is injective.
Now suppose $c \in f(X) \cap f(Y)$.
Then $c \in f(X)$ and so there exists $x \in X$ with $c = f(x)$.
And $c \in f(Y)$ so there exists $y \in Y$ with $c = f(y)$.
As $f(x) = f(y)$ and f is injective, we get $x = y$.
Since $x \in X$ and $y \in Y$ and $x = y$, we have $x \in X \cap Y$.
Now $c = f(x) \in f(X \cap Y)$.
This proves that $f(X) \cap f(Y) \subseteq f(X \cap Y)$ for f injective.
Combining this with (b) gives $f(X \cap Y) = f(X) \cap f(Y)$ for injective functions f.

Example:



Set $X = \{x, z\}$
$Y = \{y, z\}$.

Then $X \cap Y = \{z\}$
$f(X \cap Y) = f(\{z\}) = \{3\}$
$f(X) = \{1, 3\}$
$f(Y) = \{1, 3\}$
$f(X) \cap f(Y) = \{1, 3\}$