Lecture Plan Monday February 29: Counting II a

1) Counting functions

Let A and B be finite sets, with $|A| = m$ and $|B| = n$

Remark: By Rule of product, we can create n^m different functions from A to B:
If $A = \{a_1, a_2, \dots, a_m\}$, we have n choices for $f(a_1)$, n choices for $f(a_2)$, and so on.

Example: $A = \{x, y, z\}$ $B = \{0, 1\}$ $2^3 = 8$

$f_1 = \{(x,0), (y,0), (z,0)\}$
$f_2 = \{(x,0), (y,0), (z,1)\}$
$f_3 = \{(x,0), (y,1), (z,0)\}$
$f_4 = \{(x,0), (y,1), (z,1)\}$
$f_5 = \{(x,1), (y,0), (z,0)\}$
$f_6 = \{(x,1), (y,0), (z,1)\}$
$f_7 = \{(x,1), (y,1), (z,0)\}$
$f_8 = \{(x,1), (y,1), (z,1)\}$

Remark: Counting the injective functions from A to B comes down to counting permutations of size m for n objects:
If $A = \{a_1, \dots, a_m\}$ we have n choices for $f(a_1)$, then $n-1$ choices for $f(a_2)$, and then $n-2$ choices for $f(a_3)$ and so on.
So the number of injective functions from A to B is:

If $m > n$, ther are no injective functions from A to B

$$n(n-1)(n-2)\cdots(n-m+1) = \frac{n!}{(n-m)!} = P(n,m) = P(|B|, |A|)$$

Example: There are no injective functions from $\{x, y, z\}$ to $\{0, 1\}$.
There are $3 \cdot 2 = 6$ injective functions from $\{0, 1\}$ to $\{x, y, z\}$

Remark: We can also coun extensions of functions.
Suppose $X \subseteq A$ with $|X| = k$ and $g : X \to B$ is a function.
How many extensions of g from X to A are there?

We must give images to all the points in $A \setminus X$. There are $m-k$ such points.
Each point has $|B| = n$ possible images.
By Rule of product, there are n^{m-k} possible extensions.

Example: $A = \{1, 2, 3, 4, 5\}$, $B = \{x, y, z\}$, $X = \{2, 3, 4\}$
$g : X \to B$ given by $g = \{(2,y), (3,z), (4,y)\}$

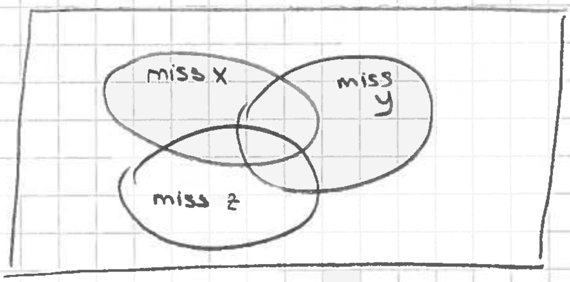
Any extension of g must have the same images for 2, 3 and 4.
There are 3 possible images for 1, and 3 possible images for 5.
This gives $3 \cdot 3 = 9$ possible extensions of g to A.

Remark: Let's try to count surjective functions next. This is a bit harder, so we start with some examples

Example: If $A = \{1,2,3,4,5\}$ and $B = \{x,y\}$, there are $2^5 = 32$ functions from A to B.
The only ones that are not surjective, are the constant ones.
So of the 32 functions, 2 functions are not surjective (the function that sends everything to x and the function that sends everything to y).
We have 30 surjective functions from A to B.

Example: If $A = \{1,2,3,4,5\}$ and $B = \{x,y,z\}$, there are 3^5 functions from A to B.

How many of these do not have x in their range?
Exactly 2^5 since all the images must now by y's and z's.
Similarly, 2^5 of the functions do not have y in their image and 2^5 functions do not have z in their image.



Next, we need to count the functions that miss both x and y. There is only 1 such function
Similarly, there is 1 function missing both y and z, and 1 function missing both x and z.
And there are no functions with empty range.

$\Rightarrow$ There are $3^5 - 2^5 - 2^5 - 2^5 + 1 + 1 + 1 = \binom{3}{3} \cdot 3^5 - \binom{3}{2} 2^5 + \binom{3}{1} 1^5$ surjective functions from A to B.

Remark: We've used the principle of inclusion/exclusion here, so let's use that more formally to see what is happening in the general case.

Setting: A set with $|A| = m$, $A = \{a_1, \dots, a_m\}$ and $m \geq 1$
B set with $|B| = n$, $B = \{b_1, \dots, b_n\}$
S = the set of all functions from A to B. So $|S| = n^m =: N$
c_i is a condition: $f \in S$ satisfies c_i if b_i is not in the range of f.

(we are interested in $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3} \cdots \overline{c_n})$)

By the principle of inclusion-exclusion, this can be calculated by:

$$N(\overline{c_1}\,\overline{c_2} \cdots \overline{c_n}) = N - \sum_{1 \leq i \leq n} N(c_i) + \sum_{1 \leq i < j \leq n} N(c_i c_j) - \sum_{1 \leq i < j < k \leq n} N(c_i c_j c_k) + \cdots \cdots + (-1)^n N(c_1 c_2 \cdots c_n)$$

So we need to calculate $N(c_i)$, $N(c_i, c_j)$ and so on.

$N(c_i) = \#\{$functions from A to B that do not have b_i in their range$\}$

$= (n-1)^m$ since we now have $n-1$ possible images for each element of A.

$N(c_i c_j) = \#\{$functions from A to B that do not have b_i or b_j in their range$\}$

$= (n-2)^m$ as long as $i \neq j$

$N(c_i c_j c_k) = (n-3)^m$ for i, j and k distinct.

and so on.....

$N(c_1 c_2 \cdots c_n) = \#\{$functions from A to B without any images$\} = 0 = (n-n)^m$

So now: $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3} \cdots \overline{c_n}) = N - \binom{n}{1} \cdot (n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m + \cdots + (-1)^n \cdot \binom{n}{n}(n-n)^m$

$$= \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m$$

$$= \sum_{i=0}^{n} (-1)^i \binom{n}{n-i} (n-i)^m$$

Remark: If $m < n$, there are no surjective functions from A to B.
So in that case, we must have

$$\sum_{i=0}^{n} (-1)^i \binom{n}{n-i} (n-i)^m = 0 \qquad \text{if } m < n$$

2) Some counting examples from earlier chapters

Example: § 3.1 example 12

For integers n and r with $1 \le r \le n$:

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

Proof 1:

$$\binom{n+1}{r} = \#\{\text{subsets of } \{1,2,3,\dots,n,n+1\} \text{ with } r \text{ elements}\}$$

$= \#\{$subsets of $\{1,\dots,n+1\}$ with r elements that contain the element n+1$\}$

$+$

$\#\{$subsets of $\{1,\dots,n+1\}$ with r elements that do not contain the element n+1$\}$

$= \#\{$subsets of $\{1,2,\dots,n\}$ with r-1 elements$\}$ (since we can add "n+1" to all those subsets later)

$+$

$\#\{$subsets of $\{1,2,\dots,n\}$ with r elements$\}$ (since those do not contain n+1 anyway)

$$= \binom{n}{r-1} + \binom{n}{r}$$

Proof 2:

$$\binom{n+1}{r} = \frac{(n+1)!}{r!\,(n+1-r)!}$$

$$\binom{n}{r} + \binom{n}{r-1} = \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-(r-1))!} = \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n+1-r)!}$$

$$= \frac{n!}{r!\,(n+1-r)!}\cdot(n+1-r) + \frac{n!}{r!\,(n+1-r)!}\cdot r = \frac{(n+1)\cdot n!}{r!\,(n+1-r)!} = \frac{(n+1)!}{r!\,(n+1-r)!}$$