~~[illegible]~~:

1) Office hours 15:00 – 17:00 10m seminar

2) Permutations and Arrangements

Def: a linear Arrangement is a sequence of objects. (so order is important)

The word INFORMATICS is a linear arrangement of letters

Def. a Permutation is a sequence of distinct objects

The word COMPUTER is a permutation, while INFORMATICS is not.

Example: Choose 5 students for a folder photo (standing in a line)

Def: $p(n,r) :=$ the number of permutations of size r for n objects.

$$= n(n-1) \cdots (n-(r-1)) = n!/(n-r)!$$

Def: For an integer $n \geq 0$, we define $n!$ (pronounce. n factorial) by:

$$n! := n(n-1) \cdots 2 \cdot 1$$
$$0! := 1$$

$0! = 1 \quad 1! = 1 \quad 2! = 2 \quad 3! = 6 \quad 4! = 24 \quad 5! = 120 \quad 6! = 720$

Example: Permutations of the word COMPUTER : $p(8,8) = 8!/0! = 8!$

Permutations of size r from the English alphabet: $p(26,8) = \frac{26!}{18!}$

Example: Arrangements of the word BALL

- BAL_1L_2 has $4!$ permutations.
- Each arrangement of BALL corresponds to 2 of these permutations (one with L_1 first, one with L_2 first)

ALBL — AL_1BL_2, AL_2BL_1

- 2 × (number of arrangements of BALL) = (number of permutations of BAL_1L_2) =4
- (number of arrangements of BALL) = $\frac{4!}{2} = 12$

Example: DATABASES 3A's, 2S's

How many permutations of $DA_1TA_2BA_3S_1ES_2$ correspond to each linear arrangement of DATABASES?

- We have S_1 and S_2: they have $2! = 2$ possible orders.
- We have A_1, A_2 and A_3: they have $3! = 6$ possible orders

⇒ For an arrangement like DAAATSBSE, we have 2 ways to assign labels to the S's and 6 to assign labels to the A's, so we find $2 \times 6 = 12$ different permutations, (by rule of product).

Hence we get $9!/12$ # arrangements

Theorem: If we have n objects of r different types, with
n_1 objects of type 1
n_2 of type 2
⋮
n_r objects of type r
~~and~~ where objects of the same type are indistinguishable
and $n_1 + n_2 + \dots + n_r = n$,

then there are $\dfrac{n!}{n_1!\, n_2! \dots n_r!}$ linear arrangements of those objects.

Example: Table seatings at a round table (2 ways)
MFMFMF table seatings.

3) Combinations: when we do not care about order

Example: Earlier today: 5 students in a row for a picture
But what if we do not care about the order, only about which students? The $5! = 120$ different orderings are now the same.

Selecting 5 out of 100 students can be done in

$$\frac{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{100!}{95! \cdot 5!}$$

ways if we do not care about order.

Notation. The number of combinations of size r from a collection of size n is

$$\binom{n}{r} = C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!} \qquad 0 \le r \le n$$

$\binom{n}{r} = C(n,r) = 0$ when $r > n$.

Example: Select 2 girls & 3 boys for the picture.
$\binom{30}{2}$ ways for the girls
$\binom{70}{3}$ ways for the guys
$\rightsquigarrow$ Rule of product: $\binom{30}{2} \cdot \binom{70}{3}$ possibilities

Example: Select at least one girl and at least 1 guy:
- 1 f 4 m: ..
- 2 f 3 m: ....
- 3 f m: ....
- 4 f 1 m: ....

__________ +

Rule of sum

Example: At least one girl:
- 5 f : ..... add also

or: $\binom{100}{5} - \binom{70}{5}$

Wrong method: $\binom{30}{1} \cdot \binom{99}{4}$: we count things double !?

Example: Number of arrangements of TALLAHASSEE: $\frac{11!}{3!2!2!2!1!1!}$

How many of those have no adjacent A's?

- First arrange only the other 8 letters: $\frac{8!}{2!2!2!1!1!}$ ways.
- T H L L E S S E ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ — 9 spots where you can place a letter A (only one).
- choose 3 of the 9 spots to ~~place~~ place A's: $\binom{9}{3}$

Notation: A way to concisely write sums.

$$4+5+6+7+8+9+10 = \sum_{k=4}^{10} k$$

$$\sum_{k=2}^{7} (k-1)^2 = (2-1)^2 + (3-1)^2 + (4-1)^2 + (5-1)^2 + (6-1)^2 + (7-1)^2$$

$$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$

$$= \sum_{i=1}^{6} i^2$$

$$= \sum_{i=0}^{6} i^2$$

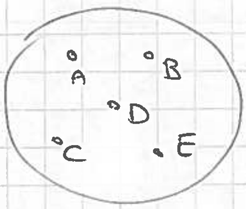
$$\sum_{k=-3}^{1} a_k = a_{-3} + a_{-2} + a_{-1} + a_0 + a_1$$

$$\binom{30}{1}\binom{70}{4} + \binom{30}{2}\binom{70}{3} + \binom{30}{3}\binom{70}{2} + \binom{30}{4}\binom{70}{1} = \sum_{k=1}^{4} \binom{30}{k}\binom{70}{5-k}$$

4) Some theorems using binomials:

Remark: For $0 \le r \le n$: $\binom{n}{r} = \binom{n}{n-r}$

Choosing which r objects to take, means choosing n−r objects to leave.



Pick two objects: $\frac{5\cdot 4}{2\cdot 1} = 10$ ways

Leave 3 behind: $\frac{5\cdot 4\cdot 3}{3\cdot 2\cdot 1} = 10$ ways

Theorem: The binomial theorem

x, y variables, n positive integer

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \dots + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

"**Proof**": First an example

$$(x+y)^2 = (x+y)(x+y) = x \cdot x + x \cdot y + y \cdot x + y \cdot y$$

The terms in this sum come from choosing either x or y in each $(x+y)$.

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$$

To get xy^3, you must choose 1 x and 3 y's. This can be done in $\binom{4}{1}$ ways.

In general: the number of "$x^k y^{n-k}$"'s in $(x+y)^n$ is $\binom{n}{k}$.

Example: We can now calculate something like $(4x-2y)^5$ as well:

$$(4x-2y)^5 = \left((4x) + (-2y)\right)^5 = \sum_{k=0}^{5} \binom{5}{k} (4x)^k (-2y)^{5-k}$$

$$= \sum_{k=0}^{5} \binom{5}{k} 4^k x^k (-2)^{5-k} \cdot y^{5-k}$$

$\Rightarrow$ The coefficient for x^3y^2 is: $\binom{5}{3} 4^3 (-2)^2 = 10 \cdot 64 \cdot 4$

Corollary: For n a positive integer:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = \binom{n}{0} \cdot 1^0 \cdot 1^n + \binom{n}{1} 1^1 \cdot 1^{n-1} + \dots + \binom{n}{n} \cdot 1^n 1^0$$

$$= (1+1)^n = 2^n$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \dots + (-1)^n \binom{n}{n} = \binom{n}{0}(-1)^0 1^n + \binom{n}{1}(-1)^1 1^{n-1} + \dots$$

$$= (-1+1)^n = 0^n = 0$$

Theorem: For positive integers n and t, the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + \dots + x_t)^n$ is

$$\frac{n!}{n_1! \, n_2! \cdots n_t!} =: \binom{n}{n_1, n_2, \dots, n_t}$$

where all n_i are integers with $0 \leq n_i \leq n$ and $n_1 + n_2 + \dots + n_t = n$

Proof: skipped.

5) Combinations with repetition:

Example: 20 Easter eggs, 5 colours.
In how many ways can I give the (identical) eggs one colour each?

So I need to distribute 20 eggs over 5 boxes.
Or, in other words, draw 5 boxes around my 20 eggs.

oo|ooooo|ooooooo||oooooo

⇝ actually, I need to order 20 eggs and 4 box dividers.

This gives us $\dfrac{(20+4)!}{20!\,4!}$ linear arrangements $\approx \binom{24}{4} = \binom{24}{20}$

Result: When we wish to select, with repetition, r of n distinct objects, then we actually consider arrangements of r objects of type A (eggs) and $n-1$ objects of type B (box dividers).
Hence, this can be done in

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!\,(n-1)!} \text{ ways}$$

Example: Suppose that in our Easter egg example, we want at least 2 blue eggs, 1 green, 1 red and 1 yellow.
Then we paint those eggs first.
Now we have 15 eggs left to colour. This can be done in $\binom{15+4}{4}$ ways.

Example: Determine all integer solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 7 \qquad \text{where } x_i \geq 0 \text{ for } i = 1,2,3,4$$

For example: $x_1 = 3,\ x_2 = 1,\ x_3 = 3,\ x_4 = 0$ or $x_1 = 1,\ x_2 = 3,\ x_3 = 0,\ x_4 = 3$
(not the same)

This comes down to distributing 7 "1"'s (or eggs) among 4 x's (or colours).

Hence: $\binom{7+4-1}{4-1} = \binom{10}{3}$ options.

Remark: The number of integer solutions of $x_1 + \dots + x_n = r$ with $x_i \geq 0$ $i = 1, \dots, n$

=

The number of selections, with repetition, of size r from a collection of size n

=

The number of ways r identical objects can be distributed among n distinct containers

=

$$\binom{r+n-1}{n-1} = \binom{r+n-1}{r}$$

Example: Integer solutions for $x_1+x_2+x_3+x_4+x_5+x_6 < 10$, $x_i \geq 0$ $i=1,\dots,6$

↳ Case by case works, but tedious.

It is also the number of integer solutions for $x_1+\cdots+x_6+x_7 \overset{=}{\cancel{\leq}} 9$ $x_i \geq 0$

which is $\binom{9+7-1}{7-1} = \binom{15}{6} = 5005$

Example: Integer solutions for $x_1+x_2+\dots+x_6 < 10$ $x_1 \geq -2$, $x_2 > 1$, $x_3 > 5$

$x_4 \geq -3$ $x_5 \geq -2$ $x_6 \geq 0$

write $y_1 = x_1+2$ (so $y_1 \geq 0$)

$y_2 = x_2-2$

$y_3 = x_3-6$

$y_4 = x_4+3$

$y_5 = x_5+2$

$y_6 = x_6$

Then we want integer solutions for

$$(y_1-2)+(y_2+2)+(y_3+6)+(y_4-3)+(y_5-2)+y_6 < 10$$

$$y_1+y_2+\dots+y_6+1 < 10$$

$$y_1+y_2+\dots+y_6 \leq 8$$

And hence for $y_1+y_2+\dots+y_6+y_7 \leq 8$ with $y_i \geq 0$

which has $\binom{8+7-1}{7-1} = \binom{14}{6}$ solutions.

Example: How many different terms does $(w+x+y+z)^{10}$ have?

Each term has the form $\binom{10}{n_1,n_2,n_3,n_4} w^{n_1}x^{n_2}y^{n_3}z^{n_4}$

So we need integer solutions for $n_1+n_2+n_3+n_4 = 10$, $(n_i \geq 0$ $i=1,\dots,4)$

There are $\binom{10+4-1}{4-1} = \binom{13}{3}$ solutions

Example: Compose 7 into summands, order counts.

↳ cases (7)

↳ other way: "start new number?"

Example:

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for i=1 to 20 do
    for j=1 to i do
        for k=1 to j do
            print (i*j+k)
```

How many prints?

$\binom{22}{3}$

Example runs of barstools