Lecture plan Mandag 25 Januari 2016 : § 2.5, § 1.1, (§ 1.2)

1) Introduction, office hours Ma 14:00 – 16:00
Please ask questions (me, Magnus, TA's)

2) Last week: logic. Proofs could be checked by a computer. p 78

3) Definitions are usually sloppy in their use of implications:

"$p \to q$", "$p \leftrightarrow q$" and "$q \to p$" are not the same.

However, in definitions we often use "$p \to q$" when we mean "$p \leftrightarrow q$".

Def: An integer n is called even if $n = 2k$ for some integer k.

Meaning:

Universe: Integers
$\forall n\, [\, n \text{ even} \leftrightarrow \exists k\, [n = 2k]\,]$

You may use this unfortunate convention only in definitions.

4) Quantified statements as premises.

Example: All goats have horns. Billy is a goat. Hence, Billy has horns

Universe: all animals
$p(x)$: x is a goat
$q(x)$: x has horns
b: the animal Billy (a specific element of the universe)

(1)	$\forall x\, [p(x) \longrightarrow q(x)]$	premise
(2)	$p(b)$	premise
(3)	$p(b) \longrightarrow q(b)$	(1) and Rule of universal specification
(4)	$\therefore q(b)$	(2), (3) and Modus Ponens.

The Rule of Universal Specification:
If "$\forall x\, p(x)$" is true, then "$p(a)$" is true for each a in the universe.

Example: All goats and all unicorns have horns.
Billy does not have a horn
So Billy is not a goat.

Universe: all animals, real and imaginary
$p(x)$: x is a goat
$q(x)$: x is a unicorn
$r(x)$: x has a horn
b: the animal Billy.

(1)	$\forall x\, [(p(x) \vee q(x)) \to r(x)]$	premise
(2)	$\neg r(b)$	premise
(3)	$(p(b) \vee q(b)] \to r(b)$	(1) and Universal Spec.
(4)	$\neg r(b) \to \neg(p(b) \vee q(b))$	(3) and contrapositive
(5)	$\neg(p(b) \vee q(b))$	(2), (4) Modus Ponens
(6)	$\neg p(b) \wedge \neg q(b)$	Rewrite (5) by DeMorgan
(7)	$\therefore \neg p(b)$	Conjunctive Simplification

Wrong example: All goals have horns
Billy has a horn.
So Billy is a goat

$\forall x : [p(x) \rightarrow r(x)]$
$r(b)$

See exercise 10 for dealing with $\exists$ as a premise.

5) Quantified statements and proving them

Example: The method of exhaustion: just check the statement for every element in the universe.

Universe: 2, 4, 6, 8, 10, 12
For all n, we can write n as a sum of at most 3 perfect squares

$$\begin{aligned}
2 &= 1+1 = 1^2+1^2 \\
4 &= 2^2 \\
6 &= 4+1+1 = 2^2+1^2+1^2 \\
8 &= 4+4 = 2^2+2^2 \\
10 &= 9+1 = 3^2+1^2 \\
12 &= 4+4+4 = 2^2+2^2+2^2
\end{aligned}$$

This only works in small universes!

Example:

$$\begin{array}{l}
\forall x\ [p(x) \rightarrow q(x)] \\
\forall x\ [q(x) \rightarrow r(x)] \\
\hline
\therefore\ \forall x\ [p(x) \rightarrow r(x)]
\end{array}$$

all goats have horns
all animals with horns are dangerous
all goats are dangerous

(1)	$\forall x\ [p(x) \rightarrow q(x)]$	premise
(2)	$p(c) \rightarrow q(c)$	c arbitrary element in universe, Rule of Univ. Sp. on (1)
(3)	$\forall x\ [q(x) \rightarrow r(x)]$	premise
(4)	$q(c) \rightarrow r(c)$	Rule of Universal Specification on (2)
(5)	$p(c) \rightarrow r(c)$	(2), (4), Law of Syllogism
(6)	$\therefore\ \forall x\ [p(x) \rightarrow r(x)]$	

The Rule of Universal Generalization:
If the open statement $p(x)$ is proved to be true when x is replaced by any arbitrarily chosen element c from our universe, then "$\forall x\, p(x)$" is true.

Similarly, if the open statement $q(x,y,z)$ is proven to be true when x, y and z are replaced by any arbitrarily chosen c, d and e from the universe, then "$\forall x \forall y \forall z\ q(x,y,z)$" is true.

Example: $\forall x\,[p(x) \vee q(x)]$
$\forall x\,[(\neg p(x) \wedge q(x)) \rightarrow r(x)]$

$\therefore \forall x\,[\neg r(x) \rightarrow p(x)]$

Use c as arbitrary element from the universe

(1)	$\forall x\,[p(x) \vee q(x)]$	premise
(2)	$p(c) \vee q(c)$	Rule of Univ. Spec. & (1)
(3)	$\forall x\,[(\neg p(x) \wedge q(x) \rightarrow r(x)]$	premise
(4)	$(\neg p(c) \wedge q(c)) \rightarrow r(c)$	Rule of Univ. Spec. & (3)
(5)	$\neg r(c) \rightarrow \neg(\neg p(c) \wedge q(c))$	Contrapositive of (4)
(6)	$\neg r(c) \rightarrow p(c) \vee \neg q(c)$	(5) and DeMorgan and double negation
(7)	$\neg r(c)$	premise (assumed)
(8)	$p(c) \vee \neg q(c)$	Modus Ponens on (6) and (7)
(9)	$(p(c) \vee q(c)) \wedge (p(c) \vee \neg q(c))$	(2) and (8) and Rule of Conjunction
(10)	$p(c) \vee (q(c) \wedge \neg q(c))$	(9) and distributive law
(11)	$p(c) \vee F_0$	$q(c) \wedge \neg q(c) \Leftrightarrow F_0$ and (10)
(12)	$p(c)$	$p(c) \vee F_0 \Leftrightarrow p(c)$
(13)	$\therefore \forall x\,[\neg r(x) \rightarrow p(x)]$	

7) Text style proof examples

Definition: Let n be an integer. We call n even if $n = 2r$ for some integer r. If n is not even, we call it odd.

Remark: If n is odd, then there is an integer s such that $n = 2s+1$.

Theorem: For all integers k and l: if k and l are both odd, then $k+l$ is even

Proof: Since k is odd, we may write $k = 2s+1$ for an integer s.
arb
$s \neq t$
univ. sp.
Since l is odd, we may write $l = 2t+1$ for some integer t.
Now $k+l = (2s+1)+(2t+1) = 2(s+t+1)$.
Since s and t are integers, $s+t+1$ is an integer.
Write $r = s+t+1$, then r is an integer and $k+l = 2r$
so $k+l$ is even.

Theorem: If m is an even integer, then $m+7$ is odd

Proof: Direct: Since m is even, $m = 2a$ for some integer a.
Then $m+7 = 2a+7 = 2(a+3)+1$. Since $a+3$ is an integer, $m+7$ is odd.

Contrapositive: If $m+7$ is not odd, then m is not even.

Suppose that $m+7$ is not odd. Then $m+7$ is even, so $m+7 = 2b$ for some integer b. Hence $m = (m+7)-7 = 2b-7 = 2(b-4)+1$. Since $b-4$ is an integer, m is odd. So m is not even. This proves the contrapositive, hence the original

Proof by contradiction:
Suppose m is even and $m+7$ is not odd, so also even.
Then $m+7 = 2c$ for some integer c, and hence $m = 2c-7$
$= 2(c-4)+1$. As $c-4$ is an integer ...

Aftenles tellen:

- Rule of sum: If task A can be performed in m ways and task B in n ways, then performing either task A or task B can be done in m+n ways

  - yellow & purple dice : 12 distinct options
  - library French / German
    Venn-diagram
  - library sociology / psychology
    Venn diagram

- Rule of product: If task A can be performed in m ways and task B in n ways, then perfo ming both task A and task B can be do in mn ways.

  - yellow & purple dice: 36 distinct options   p2 & y3 ≠ p3 & y2 !
  - library French / German
    tree-diagram
  - license plates   letter - letter - number - number - number - number
    - only vowels & even
    - no repetition

- Combination example

  - drinks & donuts / sandwich

Now continue on idea with "no repetition allowed"

- linear arrangement: ~~ordered~~ some objects placed in some order   (linear arrangement of letters
  permutation: some distinct objects placed in some order

  - choose 5 students for a folder photo (standing in a line)
  - Definition: $P(n,r) :=$ the number of permutations of size r for n objects.

    $$= n \cdot (n-1) \cdot \ldots \cdot (n-(r-1))$$

  - Definition: For an integer $n \geq 0$ we can define n factorial by:
    $n! := n(n-1)(n-2) \cdots 1$
    $0! := 1$

    $0!$ $1!$ $2!$ $3!$ $4!$ $5!$ $6!$

  - $P(n,r) = \frac{n!}{(n-r)!}$

- Permutations of the word computer + all letters are distinct.