① §5.7 3.26

Recall • g dominanents f if

$$|f(x)| \le m \cdot |g(x)| \quad \text{for some } m \in \mathbb{R}^+ \ \& \ \forall x \ge k (\in \mathbb{Z}^+)$$

(Write $f \in O(g)$).

• $f \sim g$ if $f \in O(g)$ & $g \in O(f)$

$O(f) < O(g)$ if $f \in O(g)$ but $g \notin O(f)$.

So: the comon Big-Oh forms are:

$$O(1) < O(\log_2 x) < O(x) < O(x^2) < \cdots < O(x^n) < O(e^x) < O(x!)$$

Ex: $f_t(x) = 1^t + 2^t + \cdots + x^t$

$= \sum_{i=1}^{t} x^t$

$\sum_{i=\frac{t}{2}}^{t} (\cancel{\frac{x}{2}})^2 \le f_t(x) \le x^t + \cdots + x^t = \sum_{i=1}^{t} x^t$

$t \ge 0 (\in \mathbb{Z})$ $f_t(x) \sim x^{t+1}$ $= \frac{x^{t+1}}{2^{t+1}}$ $= x^{t+1}$

$\Rightarrow x^{t+1} \in O(f_t(x))$ & $f_t(x) \in O(x^{t+1})$

$t = -1$? $f_t(x) = 1 + \frac{1}{2} + \cdots + \frac{1}{x}$

Claim $f_t(x) \sim \log_2 x$ (in fact $\lim_{x \to \infty} f_t(x) - \ln x = 0.5\ldots$)

$$f_t(x) = 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{15}\right) + \cdots$$

So $f_t(x) \le 1 + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) + \cdots$

$\le 1 + 1 + 1 + \cdots$

$\le \lceil \log_2 x \rceil$

$f_t(x) \ge \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots$

$\ge \lfloor \log_2 x \rfloor$

②

Thus $f_t(x) \in \Theta(\log_2 x)$

& $\log_2 x \in \Theta(f_t(x))$ $\boxed{///}$

§5.8 Analysis of Algorithms

Ex 5.69 Account Balance $n \in \mathbb{Z}^+$.

Begin

initialization $\left\{ \begin{array}{l} D = \text{deposite} = 50. \\ i = 1 \\ R = \text{rate} = 0.005 \end{array} \right.$ (monthly)

$B = \text{balance} = 100$

while $i \leq n$ do

$B = D + B + B * R$; $i = i + 1$

End

- The task: Count/measure the total number of operations

$f(n) = \#\text{operations}$. (s.t. assignments + * comparisons ..)

- initialization: 4 ops.
- In each loop 7 ops ($\leq$ = + + * = +)
- At the end when $i = n+1$, one more comparison.

In total: $f(n) = 4 + 7n + 1 \sim \Theta(n)$.

The point: #loops provides the shortcut answer (when in the loop, $\exists$ const ops).

initial & end "does not matter".

③

Ex5.7. Search process:

- $\exists$ a sequence of integers: $a_1 \cdots a_n$.
- Search for an integer (the key).

Output: the location of the key, if found

$= 0$ if not found.

Algorithm Linear seach

Begin

$i = 1$

Loop [while ($i \le n$ & key $\ne a_i$) do

$i = i + 1$

if $i \le n$ then location $= i$ ✓

else location $= 0$ ×

end.

Analysis Best case: key $= a_1$ $f(n) = \text{const} \sim O(1)$

Worst case: key not found $f(n) \sim O(n)$ n loops

Expectation: Suppose $\text{Prob}(\text{key} = a_i) = p$

& $q = 1 - np = \text{Prob}(\text{key not found})$.

Then $f(n) = \left(\frac{1}{p} + \frac{2}{p} + \cdots + \frac{n}{p}\right) \cdot \text{const} + \frac{n}{q} \cdot \text{const}$. (can take const $= 1$)

$= \frac{p}{2} n(n+1) + nq$.

- If $p = \frac{1}{n}$ then $\sim O(n)$.
  ($q = 0$)

If q is fixed, then $p = \frac{1-q}{n} \Rightarrow f(n) \sim O(n)$.

④

Ex5.72 Calculate a^n.

Alg1 Begin

$x=1$

for $i=1$ to n

$x=x*a$

End

$f(n)=n \sim O(n)$.

Ex5.73

Alg2

Begin

$x=1$ $i=n$

while $i>0$ do

If $i \neq 2*\lfloor\frac{i}{2}\rfloor$ (i.e. $2 \nmid i$) then

$x=x*a$

$i=\lfloor\frac{i}{2}\rfloor$

If $i>0$ then

$a=a*a$

End

End

$g(n) \sim \log_2 n$.

a^7: $\begin{matrix} x=1 \\ i=7 \end{matrix} \rightsquigarrow \begin{matrix} x=a \\ i=3 \\ \underline{a}=a^2 \end{matrix} \rightsquigarrow \begin{matrix} x=a^3 \\ i=1 \\ \underline{a}=a^4 \end{matrix} \rightsquigarrow x=a^7$

a^8 $\begin{matrix} x=1 \\ i=8 \end{matrix} \rightsquigarrow \begin{matrix} x=1 \\ i=4 \\ \underline{a}=a^2 \end{matrix} \rightsquigarrow \begin{matrix} x=1 \\ i=2 \\ \underline{a}=a^4 \end{matrix} \rightsquigarrow \begin{matrix} x=1 \\ i=1 \\ \underline{a}=a^8 \end{matrix} \rightsquigarrow x=a^8$

$\underline{a}$ is the variable

a is the original value

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If $\lceil \log_2 n \rceil = h \Rightarrow n \leq 2^h$

$\Rightarrow$ after h loops $i \leq \frac{2^h}{2^h} \leq 1$

$\Rightarrow g(n) \leq h+1$.

$\Rightarrow g(n) \in O(\log_2 n)$

If $\lfloor \log_2 n \rfloor = h' \Rightarrow 2^{h'} \leq n$

$\Rightarrow$ after h' loops $1 = \frac{2^{h'}}{2^{h'}} \leq i$

$\boxed{y \leq x \Rightarrow \lfloor \frac{y}{2} \rfloor \leq \lfloor \frac{x}{2} \rfloor}$

$\Rightarrow g(n) \geq h'+1$

$\Rightarrow \log_2 n \in O(g)$

Thus $g \sim O(\log_2 n)$

So $g(n) < f(n)$