① 3.24

Recall §12.3

Sorting $x_1 \cdots x_n$ ($\in \mathbb{R}$, $x_i \neq x_j$ if $i \neq j$) in ascending order.

Bubble Sort

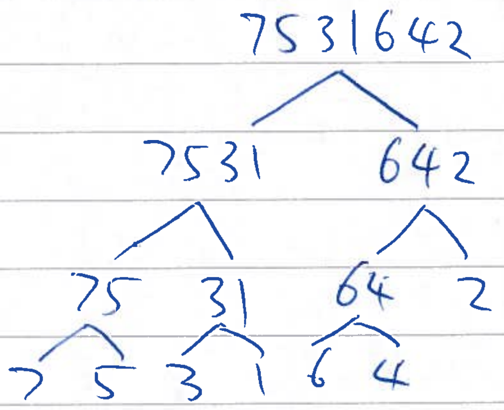
For $i=1$ to n
  For $j=n$ to $i+1$
    If $x_j < x_{j-1}$ then exchange (x_{j-1}, x_j)
  End
End

#comparisons $= \binom{n}{2}$ #exchanges $\leq \binom{n}{2}$

Time Complexity $f(n) = \binom{n}{2}$

Merge Sort

Step 1. Splitting the List $(x_1 \cdots x_n)$ into n lists (each of which contains a single number) s.t. they form a complete binary tree.



Step 2. Merge them back:

at each vertex. $L_{ij} = M(L_i, L_j)$ merge 2 ascending lists into 1 ascending list

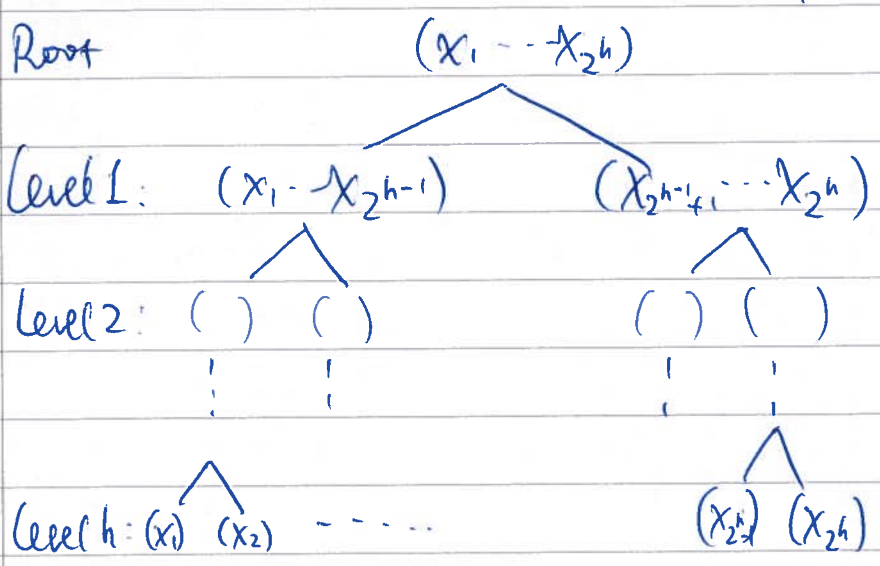
"Like a team play"

$a_k \cdots a_3 a_2 a_1$
"versus" $\min(a_1, b_1)$
$b_\ell \cdots b_3 b_2 b_1$

$L_{ij} =$ ↓ loser

at most $k+\ell-1$ matches

② Consider the case when $n = 2^h$ for simplicity.



#comparisons at Level $h \leq 2^{h-1}$ pairs $\times (2^1 - 1)$ comparison

at level $h-1 \leq 2^{h-2}$ pairs $\times (2^2 - 1)$ comparisons.

$\vdots$

at level $1 \leq 2^0$ pairs $\times (2^h - 1)$ comparisons

In total:

$$\sum_{k=1}^{h} 2^{k-1}(2^{h-k+1} - 1) = \sum_{k=1}^{h} 2^h - \sum_{k=1}^{h} 2^{k-1}$$
$$= h \cdot 2^h - \sum_{k=0}^{h-1} 2^k$$
$$= h \cdot 2^h - 2^h + 1$$

Since $h = \log_2 n$
$$= n\log_2 n - n + 1.$$

Ques: How to compare $\binom{n}{2}$ & $n\log_2 n - n + 1$.

§5.7 Complexity. $f, g : \mathbb{Z}^+ \to \mathbb{R}$.

Def. We say g dominants f if $\exists\, m \in \mathbb{R}^+, k \in \mathbb{Z}^+$ s.t.

$$|f(x)| \leq m\,|g(x)| \quad \forall x \geq k \ (\in \mathbb{N})$$

③

In such a case write $f \in \mathcal{O}(g)$.

If $f \in \mathcal{O}(g)$, $g \in \mathcal{O}(f)$, then we write $f \sim g$ — equivalent at growing speed.

$f \sim \mathcal{O}(g)$

$g \sim \mathcal{O}(f)$

"Big-Oh"

Ex. 1/ $f(x) = 5x \quad g(x) = x^2$

f:	5	10	15	20	25	30	35	...
g:	1	4	9	16	25	36	49	...
n =	1	2	3	4	5	6	7	

Claim: $f \in \mathcal{O}(g)$ but $g \notin \mathcal{O}(f)$.

$(m = 5, k = 1)$ (If so, $\exists m, k$, ... then take $x = 5m + k$.)

2/ $f(x) = x^2 - x \quad g(x) = 5x^2$

Claim $f \sim g$.

obv. $f \in \mathcal{O}(g)$

How about $g \in \mathcal{O}(f)$ $\quad m = 6 \quad k = 6$.

In general: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \quad (a_n \neq 0)$

Then $f(x) \sim \mathcal{O}(x^n)$.

On the one hand $|f(x)| \leq \underbrace{\left(\sum |a_i|\right)}_{= m} \cdot x^n \quad x \geq 1$.

On the other hand. $x^n \leq m|f(x)|$

$\Leftarrow \quad x^n \leq |ma_n| x^n - |ma_{n-1}| x^{n-1} - \cdots - |ma_0|$

$\Leftrightarrow \frac{|ma_{n-1}|}{x} + \frac{|ma_{n-2}|}{x^2} + \cdots \leq (|ma_n| - 1) \cancel{x^n}$

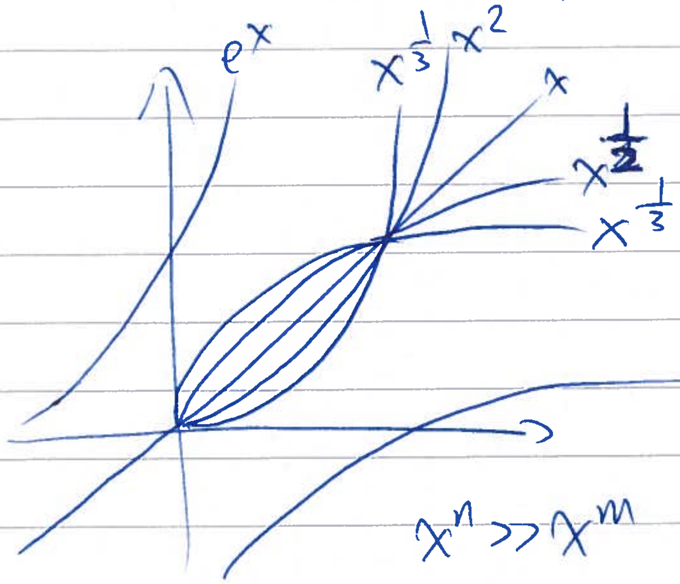
Take $k = \sum m|a_i|$ & $m = \frac{2}{|a_n|}$.

④ Big-Oh Form

$O(x\log_2 x)$

Ex: $O(1)$ const. $O(\log_2 x)$ Logarithmic $O(x)$ linear $O(x^2)$ quadratic $O(x^n)$ poly. $O(e^x)$ exponential $O(x!)$ factorial

From Graph:



$x^n >> x^m \quad n > m.$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (>> x^n)$$

$\Updownarrow$ inverse

$$\log x \qquad (<< x^n).$$

Ex. $f_t(x) = \sum_{i=1}^{x} x^t$

$$f_1(x) = 1 + 2 + \cdots + x = \frac{x(x+1)}{2} \sim O(x^2)$$

$$f_2(x) = 1^2 + 2^2 + \cdots + x^2 = \frac{x(x+1)(2x+1)}{6} \sim O(x^3)$$

Show that $\forall t \geq 1,\ t \in \mathbb{Z}^+$

$$f_t(x) \sim O(x^{t+1}).$$

Ques: How about $f_{-1}(x)$?

⑤

$$f_{-1}(x) = 1 + \frac{1}{2} + \cdots + \frac{1}{x}.$$

$$1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\cdots\right)+\cdots \leq f_{-1}(x) \leq 1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\cdots+\frac{1}{7}\right)+\left(\frac{1}{8}+\cdots\right)+$$

$$\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\cdots\right)+\cdots \leq \qquad \leq 1+\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\cdots+\frac{1}{4}\right)+$$

$$\frac{1}{2}+\frac{1}{2}+\cdots\cdots \leq \qquad \leq 1+1+1+\cdots$$

$$\frac{\log_2 x}{2} \leq \qquad \leq \log_2 x$$

$$\Rightarrow f_{-1}(x) \sim \log_2 x$$

Back to sorting $\quad \Theta(x^2) \ni \Theta(x \log_2 x).$

Lem $\quad f \in \Theta(g)$ then $f \cdot h \in \Theta(g \cdot h).$

Lem $\quad x^n \in \Theta(e^x)$ $\qquad$ fixed $N \in \mathbb{Z}^+$

$$e^x \geq 2^x = \binom{x}{0} + \binom{x}{1} + \cdots + \binom{x}{x} \geq \binom{x}{N} \sim \Theta(x^N).$$

$$\log_2 x \in \Theta(x^n). \qquad \log_2 x \leq m\, x^n$$

$$\Leftrightarrow \quad y \leq m\left(2^y\right)^n$$