

1.15

We now do something a bit more abstract and prove that if  $n$  and  $k$  are positive integers with  $n = 2k$ , then  $n!/2^k$  is an integer. Because our argument relies on counting, it is an example of a *combinatorial proof*.

Consider the  $n$  symbols  $x_1, x_1, x_2, x_2, \dots, x_k, x_k$ . The number of ways in which we can arrange all of these  $n = 2k$  symbols is an integer that equals

$$\frac{n!}{\underbrace{2! \cdot 2! \cdots 2!}_{k \text{ factors of } 2!}} = \frac{n!}{2^k}.$$

Finally, we will apply what has been developed so far to a situation in which the arrangements are no longer linear.

1.16

If six people, designated as A, B, ..., F, are seated about a round table, how many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotation? [In Fig. 1.2, arrangements (a) and (b) are considered identical, whereas (b), (c), and (d) are three distinct arrangements.]

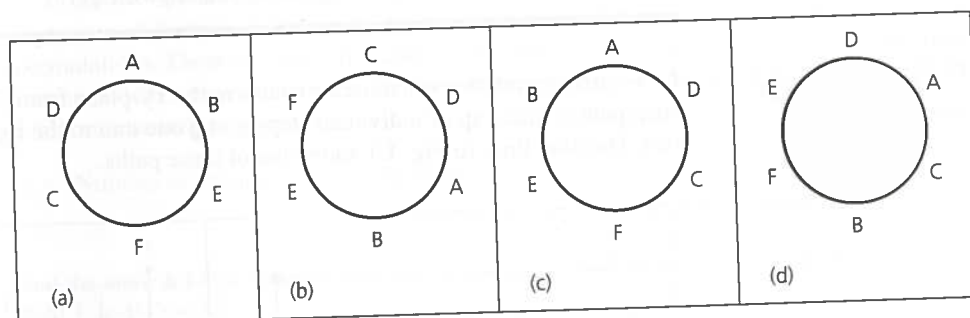


Figure 1.2

We shall try to relate this problem to previous ones we have already encountered. Consider Figs. 1.2(a) and (b). Starting at the top of the circle and moving clockwise, we list the distinct linear arrangements ABEFCD and CDABEF, which correspond to the same circular arrangement. In addition to these two, four other linear arrangements — BEFCDA, DABEFC, EFCDA B, and FCDABE — are found to correspond to the same circular arrangement as in (a) or (b). So inasmuch as each circular arrangement corresponds to six linear arrangements, we have  $6 \times (\text{Number of circular arrangements of } A, B, \dots, F) = (\text{Number of linear arrangements of } A, B, \dots, F) = 6!$ .

Consequently, there are  $6!/6 = 5! = 120$  arrangements of A, B, ..., F around the circular table.

LE 1.17

Suppose now that the six people of Example 1.16 are three married couples and that A, B, and C are the females. We want to arrange the six people around the table so that the sexes alternate. (Once again, arrangements are considered identical if one can be obtained from the other by rotation.)

Before we solve this problem, let us solve Example 1.16 by an alternative method, which will assist us in solving our present problem. If we place A at the table as shown in Fig. 1.3(a), five locations (clockwise from A) remain to be filled. Using B, C, ..., F to fill

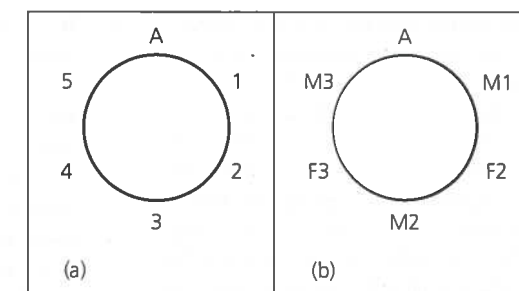


Figure 1.3

these five positions is the problem of permuting B, C, ..., F in a linear manner, and this can be done in  $5! = 120$  ways.

To solve the new problem of alternating the sexes, consider the method shown in Fig. 1.3(b). A (a female) is placed as before. The next position, clockwise from A, is marked M1 (Male 1) and can be filled in three ways. Continuing clockwise from A, position F2 (Female 2) can be filled in two ways. Proceeding in this manner, by the rule of product, there are  $3 \times 2 \times 2 \times 1 \times 1 = 12$  ways in which these six people can be arranged with no two men or women seated next to each other.

## EXERCISES 1.1 AND 1.2

1. During a local campaign, eight Republican and five Democratic candidates are nominated for president of the school board.

- If the president is to be one of these candidates, how many possibilities are there for the eventual winner?
- How many possibilities exist for a pair of candidates (one from each party) to oppose each other for the eventual election?
- Which counting principle is used in part (a)? in part (b)?

2. Answer part (c) of Example 1.6.

3. Buick automobiles come in four models, 12 colors, three engine sizes, and two transmission types. (a) How many distinct Buicks can be manufactured? (b) If one of the available colors is blue, how many different blue Buicks can be manufactured?

4. The board of directors of a pharmaceutical corporation has 10 members. An upcoming stockholders' meeting is scheduled to approve a new slate of company officers (chosen from the 10 board members).

- How many different slates consisting of a president, vice president, secretary, and treasurer can the board present to the stockholders for their approval?
- Three members of the board of directors are physicians. How many slates from part (a) have (i) a physician nominated for the presidency? (ii) exactly one physician appear-

ing on the slate? (iii) at least one physician appearing on the slate?

5. While on a Saturday shopping spree Jennifer and Tiffany witnessed two men driving away from the front of a jewelry shop, just before a burglar alarm started to sound. Although everything happened rather quickly, when the two young ladies were questioned they were able to give the police the following information about the license plate (which consisted of two letters followed by four digits) on the get-away car. Tiffany was sure that the second letter on the plate was either an O or a Q and the last digit was either a 3 or an 8. Jennifer told the investigator that the first letter on the plate was either a C or a G and that the first digit was definitely a 7. How many different license plates will the police have to check out?

6. To raise money for a new municipal pool, the chamber of commerce in a certain city sponsors a race. Each participant pays a \$5 entrance fee and has a chance to win one of the different-sized trophies that are to be awarded to the first eight runners who finish.

- If 30 people enter the race, in how many ways will it be possible to award the trophies?
- If Roberta and Candice are two participants in the race, in how many ways can the trophies be awarded with these two runners among the top three?

7. A certain "Burger Joint" advertises that a customer can have his or her hamburger with or without any or all of the following: catsup, mustard, mayonnaise, lettuce, tomato, onion, pickle, cheese, or mushrooms. How many different kinds of hamburger orders are possible?