We can rephrase Definition 3.10 by using quantifiers:

$$x \in \bigcup_{i \in I} A_i \Leftrightarrow \exists i \in I (x \in A_i) \qquad x \in \bigcap_{i \in I} A_i \Leftrightarrow \forall i \in I (x \in A_i)$$

Then $x \notin \cup_{i \in I} A_i \Leftrightarrow \neg [\exists i \in I (x \in A_i)] \Leftrightarrow \forall i \in I (x \notin A_i)$; that is, $x \notin \cup_{i \in I} A_i$ if and only if $x \notin A_i$ for *every* index $i \in I$. Similarly, $x \notin \cap_{i \in I} A_i \Leftrightarrow \neg [\forall i \in I (x \in A_i)] \Leftrightarrow \exists i \in I (x \notin A_i)$; that is, $x \notin \cap_{i \in I} A_i$ if and only if $x \notin A_i$ for *at least one* index $i \in I$.

If the index set I is the set $\mathbf{Z}^+$, we can write

$$\bigcup_{i \in \mathbf{Z}^+} A_i = A_1 \cup A_2 \cup \cdots = \bigcup_{i=1}^{\infty} A_i, \qquad \bigcap_{i \in \mathbf{Z}^+} A_i = A_1 \cap A_2 \cap \cdots = \bigcap_{i=1}^{\infty} A_i.$$

EXAMPLE 3.23

Let $I = \{3, 4, 5, 6, 7\}$, and for each $i \in I$ let $A_i = \{1, 2, 3, \ldots, i\} \subseteq \mathcal{U} = \mathbf{Z}^+$. Then $\cup_{i \in I} A_i = \cup_{i=3}^{7} A_i = \{1, 2, 3, \ldots, 7\} = A_7$, whereas $\cap_{i \in I} A_i = \{1, 2, 3\} = A_3$.

EXAMPLE 3.24

Let $\mathcal{U} = \mathbf{R}$ and $I = \mathbf{R}^+$. If for each $r \in \mathbf{R}^+$, $A_r = [-r, r]$, then $\cup_{r \in I} A_r = \mathbf{R}$ and $\cap_{r \in I} A_r = \{0\}$.

When dealing with generalized unions and intersections, membership tables and Venn diagrams are unfortunately next to useless, but the rigorous element approach, as demonstrated in the first part of the proof of Theorem 3.3, is still available.

THEOREM 3.6

Generalized DeMorgan's Laws. Let I be an index set where for each $i \in I$, $A_i \subseteq \mathcal{U}$. Then

a) $\overline{\bigcup_{i \in I} A_i} = \bigcap_{i \in I} \overline{A_i}$ b) $\overline{\bigcap_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}$

Proof: We shall prove Theorem 3.6(a) and leave the proof of part (b) for the reader. For each $x \in \mathcal{U}$, $x \in \overline{\cup_{i \in I} A_i} \Leftrightarrow x \notin \cup_{i \in I} A_i \Leftrightarrow x \notin A_i$, for all $i \in I \Leftrightarrow x \in \overline{A_i}$, for all $i \in I \Leftrightarrow x \in \cap_{i \in I} \overline{A_i}$.

EXERCISES 3.2

1. For $\mathcal{U} = \{1, 2, 3, \ldots, 9, 10\}$ let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 4, 8\}$, $C = \{1, 2, 3, 5, 7\}$, and $D = \{2, 4, 6, 8\}$. Determine each of the following:

a) $(A \cup B) \cap C$ b) $A \cup (B \cap C)$
c) $\overline{C} \cup \overline{D}$ d) $\overline{C \cap D}$
e) $(A \cup B) - C$ f) $A \cup (B - C)$
g) $(B - C) - D$ h) $B - (C - D)$
i) $(A \cup B) - (C \cap D)$

2. If $A = [0, 3]$, $B = [2, 7)$, with $\mathcal{U} = \mathbf{R}$, determine each of the following:

a) $A \cap B$ b) $A \cup B$
c) $\overline{A}$ d) $A \triangle B$
e) $A - B$ f) $B - A$

3. a) Determine the sets A, B where $A - B = \{1, 3, 7, 11\}$, $B - A = \{2, 6, 8\}$, and $A \cap B = \{4, 9\}$.

b) Determine the sets C, D where $C - D = \{1, 2, 4\}$, $D - C = \{7, 8\}$, and $C \cup D = \{1, 2, 4, 5, 7, 8, 9\}$.

4. Let $A, B, C, D, E \subseteq \mathbf{Z}$ be defined as follows:

$A = \{2n | n \in \mathbf{Z}\}$—that is, A is the set of all (integer) multiples of 2;

$B = \{3n | n \in \mathbf{Z}\}$; $C = \{4n | n \in \mathbf{Z}\}$;

$D = \{6n | n \in \mathbf{Z}\}$; and $E = \{8n | n \in \mathbf{Z}\}$.

a) Which of the following statements are true and which are false?

i) $E \subseteq C \subseteq A$ ii) $A \subseteq C \subseteq E$
iii) $B \subseteq D$ iv) $D \subseteq B$
v) $D \subseteq A$ vi) $\overline{D} \subseteq \overline{A}$

b) Determine each of the following sets.

i) $C \cap E$ ii) $B \cup D$ iii) $A \cap B$
iv) $B \cap D$ v) $\overline{A}$ vi) $A \cap E$

5. Determine which of the following statements are true and which are false.

a) $\mathbf{Z}^+ \subseteq \mathbf{Q}^+$ b) $\mathbf{Z}^+ \subseteq \mathbf{Q}$
c) $\mathbf{Q}^+ \subseteq \mathbf{R}$ d) $\mathbf{R}^+ \subseteq \mathbf{Q}$
e) $\mathbf{Q}^+ \cap \mathbf{R}^+ = \mathbf{Q}^+$ f) $\mathbf{Z}^+ \cup \mathbf{R}^+ = \mathbf{R}^+$
g) $\mathbf{R}^+ \cap \mathbf{C} = \mathbf{R}^+$ h) $\mathbf{C} \cup \mathbf{R} = \mathbf{R}$
i) $\mathbf{Q}^* \cap \mathbf{Z} = \mathbf{Z}$

6. Prove each of the following results without using Venn diagrams or membership tables. (Assume a universe $\mathcal{U}$.)

a) If $A \subseteq B$ and $C \subseteq D$, then $A \cap C \subseteq B \cap D$ and $A \cup C \subseteq B \cup D$.

b) $A \subseteq B$ if and only if $A \cap \overline{B} = \emptyset$.

c) $A \subseteq B$ if and only if $\overline{A} \cup B = \mathcal{U}$.

7. Prove or disprove each of the following:

a) For sets $A, B, C \subseteq \mathcal{U}$, $A \cap C = B \cap C \Rightarrow A = B$.

b) For sets $A, B, C \subseteq \mathcal{U}$, $A \cup C = B \cup C \Rightarrow A = B$.

c) For sets $A, B, C \subseteq \mathcal{U}$, $[(A \cap C = B \cap C) \wedge (A \cup C = B \cup C)] \Rightarrow A = B$.

d) For sets, $A, B, C \subseteq \mathcal{U}$, $A \triangle C = B \triangle C \Rightarrow A = B$.

8. Using Venn diagrams, investigate the truth or falsity of each of the following, for sets $A, B, C \subseteq \mathcal{U}$.

a) $A \triangle (B \cap C) = (A \triangle B) \cap (A \triangle C)$

b) $A - (B \cup C) = (A - B) \cap (A - C)$

c) $A \triangle (B \triangle C) = (A \triangle B) \triangle C$

9. If $A = \{a, b, d\}$, $B = \{d, x, y\}$, and $C = \{x, z\}$, how many proper subsets are there for the set $(A \cap B) \cup C$? How many for the set $A \cap (B \cup C)$?

10. For a given universal set $\mathcal{U}$, each subset A of $\mathcal{U}$ satisfies the idempotent laws of union and intersection. (a) Are there any real numbers that satisfy an idempotent property for addition? (That is, can we find any real number(s) x such that $x + x = x$?) (b) Answer part (a) upon replacing addition by multiplication.

11. Write the dual statement for each of the following set-theoretic results.

a) $\mathcal{U} = (A \cap B) \cup (A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B})$

b) $A = A \cap (A \cup B)$

c) $A \cup B = (A \cap B) \cup (A \cap \overline{B}) \cup (\overline{A} \cap B)$

d) $A = (A \cup B) \cap (A \cup \emptyset)$

12. Let $A, B \subseteq \mathcal{U}$. Use the equivalence $A \subseteq B \Leftrightarrow A \cap B = A$ to show that the dual statement of $A \subseteq B$ is the statement $B \subseteq A$.

13. Prove or disprove each of the following for sets $A, B \subseteq \mathcal{U}$.

a) $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$

b) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$

14. Use membership tables to establish each of the fol

a) $\overline{A \cap B} = \overline{A} \cup \overline{B}$ b) $A \cup A = A$

c) $A \cup (A \cap B) = A$

d) $\overline{(A \cap B) \cup (\overline{A} \cap C)} = (A \cap \overline{B}) \cup (\overline{A} \cap \overline{C})$

15. a) How many rows are needed to construct the mem table for $A \cap (B \cup C) \cap (D \cup \overline{E} \cup \overline{F})$?

b) How many rows are needed to construct the m ship table for a set made up from the sets $A_1, A_2, .$ using $\cap$, $\cup$, and $\overline{}$?

c) Given the membership tables for two sets A, can the relation $A \subseteq B$ be recognized?

d) Use membership tables to determine whethe $(A \cap B) \cup \overline{(B \cap C)} \supseteq A \cup \overline{B}$.

16. Provide the justifications (selected from the law theory) for the steps that are needed to simplify the set

$$(A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap \overline{D}))],$$

where $A, B, C, D \subseteq \mathcal{U}$.

Steps **Re**

$(A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap \overline{D}))]$
$= (A \cap B) \cup [B \cap (C \cap (D \cup \overline{D}))]$
$= (A \cap B) \cup [B \cap (C \cap \mathcal{U})]$
$= (A \cap B) \cup (B \cap C)$
$= (B \cap A) \cup (B \cap C)$
$= B \cap (A \cup C)$

17. Using the laws of set theory, simplify each of the fol

a) $A \cap (B - A)$

b) $(A \cap B) \cup (A \cap B \cap \overline{C} \cap D) \cup (\overline{A} \cap B)$

c) $(A - B) \cup (A \cap B)$

d) $\overline{A} \cup \overline{B} \cup (A \cap B \cap \overline{C})$

18. For each $n \in \mathbf{Z}^+$ let $A_n = \{1, 2, 3, \ldots, n - 1, n$ $\mathcal{U} = \mathbf{Z}^+$ and the index set $I = \mathbf{Z}^+$.) Determine

$$\bigcup_{n=1}^{7} A_n, \quad \bigcap_{n=1}^{11} A_n, \quad \bigcup_{n=1}^{m} A_n, \quad \text{and} \quad \bigcap_{n=1}^{m} A_n,$$

where m is a fixed positive integer.

19. Let $\mathcal{U} = \mathbf{R}$ and let $I = \mathbf{Z}^+$. For each $n \in \mathbf{Z}^+$ le $[-2n, 3n]$. Determine each of the following:

a) A_3 b) A_4
c) $A_3 - A_4$ d) $A_3 \triangle A_4$
e) $\bigcup_{n=1}^{7} A_n$ f) $\bigcap_{n=1}^{7} A_n$
g) $\bigcup_{n \in \mathbf{Z}^+} A_n$ h) $\bigcap_{n=1}^{\infty} A_n$

20. Provide the details for the proof of Theorem 3.6(b)