h) $\mathbf{R}^+$ = the set of *positive real numbers*

i) $\mathbf{R}^*$ = the set of *nonzero real numbers*

j) $\mathbf{C}$ = the set of *complex numbers* $= \{x + yi \mid x, y \in \mathbf{R}, i^2 = -1\}$

k) $\mathbf{C}^*$ = the set of *nonzero complex numbers*

l) For each $n \in \mathbf{Z}^+$, $\mathbf{Z}_n = \{0, 1, 2, \ldots, n-1\}$

m) For real numbers a, b with $a < b$, $[a, b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$, $(a, b) = \{x \in \mathbf{R} \mid a < x < b\}$, $[a, b) = \{x \in \mathbf{R} \mid a \leq x < b\}$, $(a, b] = \{x \in \mathbf{R} \mid a < x \leq b\}$. The first set is called a *closed interval*, the second set an *open interval*, and the other two sets *half-open intervals*.

EXERCISES 3.1

1. Which of the following sets are equal?

a) $\{1, 2, 3\}$ b) $\{3, 2, 1, 3\}$
c) $\{3, 1, 2, 3\}$ d) $\{1, 2, 2, 3\}$

2. Let $A = \{1, \{1\}, \{2\}\}$. Which of the following statements are true?

a) $1 \in A$ b) $\{1\} \in A$
c) $\{1\} \subseteq A$ d) $\{\{1\}\} \subseteq A$
e) $\{2\} \in A$ f) $\{2\} \subseteq A$
g) $\{\{2\}\} \subseteq A$ h) $\{\{2\}\} \subset A$

3. For $A = \{1, 2, \{2\}\}$, which of the eight statements in Exercise 2 are true?

4. Which of the following statements are true?

a) $\emptyset \in \emptyset$ b) $\emptyset \subset \emptyset$ c) $\emptyset \subseteq \emptyset$
d) $\emptyset \in \{\emptyset\}$ e) $\emptyset \subset \{\emptyset\}$ f) $\emptyset \subseteq \{\emptyset\}$

5. Determine all of the elements in each of the following sets.

a) $\{1 + (-1)^n \mid n \in \mathbf{N}\}$
b) $\{n + (1/n) \mid n \in \{1, 2, 3, 5, 7\}\}$
c) $\{n^3 + n^2 \mid n \in \{0, 1, 2, 3, 4\}\}$

6. Consider the following six subsets of $\mathbf{Z}$.

$A = \{2m + 1 \mid m \in \mathbf{Z}\}$ $B = \{2n + 3 \mid n \in \mathbf{Z}\}$
$C = \{2p - 3 \mid p \in \mathbf{Z}\}$ $D = \{3r + 1 \mid r \in \mathbf{Z}\}$
$E = \{3s + 2 \mid s \in \mathbf{Z}\}$ $F = \{3t - 2 \mid t \in \mathbf{Z}\}$

Which of the following statements are true and which are false?

a) $A = B$ b) $A = C$ c) $B = C$
d) $D = E$ e) $D = F$ f) $E = F$

7. Let A, B be sets from a universe $\mathscr{U}$. (a) Write a quantified statement to express the proper subset relation $A \subset B$. (b) Negate the result in part (a) to determine when $A \not\subset B$.

8. For $A = \{1, 2, 3, 4, 5, 6, 7\}$, determine the number of

a) subsets of A
b) nonempty subsets of A
c) proper subsets of A
d) nonempty proper subsets of A
e) subsets of A containing three elements
f) subsets of A containing 1, 2
g) subsets of A containing five elements, including 1, 2
h) subsets of A with an even number of elements
i) subsets of A with an odd number of elements

9. a) If a set A has 63 proper subsets, what is $|A|$?

b) If a set B has 64 subsets of odd cardinality, what is $|B|$?

c) Generalize the result of part (b)

10. Which of the following sets are nonempty?

a) $\{x \mid x \in \mathbf{N}, 2x + 7 = 3\}$
b) $\{x \in \mathbf{Z} \mid 3x + 5 = 9\}$
c) $\{x \mid x \in \mathbf{Q}, x^2 + 4 = 6\}$
d) $\{x \in \mathbf{R} \mid x^2 + 4 = 6\}$
e) $\{x \in \mathbf{R} \mid x^2 + 3x + 3 = 0\}$
f) $\{x \mid x \in \mathbf{C}, x^2 + 3x + 3 = 0\}$

11. When she is about to leave a restaurant counter, Mrs. Albanese sees that she has one penny, one nickel, one dime, one quarter, and one half-dollar. In how many ways can she leave some (at least one) of her coins for a tip if (a) there are no restrictions? (b) she wants to have some change left? (c) she wants to leave at least 10 cents?

12. Let $A = \{1, 2, 3, 4, 5, 7, 8, 10, 11, 14, 17, 18\}$.

a) How many subsets of A contain six elements?

b) How many six-element subsets of A contain four even integers and two odd integers?

c) How many subsets of A contain only odd integers?

13. Let $S = \{1, 2, 3, \ldots, 29, 30\}$. How many subsets A of S satisfy (a) $|A| = 5$? (b) $|A| = 5$ and the smallest element in A is 5? (c) $|A| = 5$ and the smallest element in A is less than 5?

14. a) How many subsets of $\{1, 2, 3, \ldots, 11\}$ contain at least one even integer?

b) How many subsets of $\{1, 2, 3, \ldots, 12\}$ contain at least one even integer?

c) Generalize the results of parts (a) and (b).

15. Give an example of three sets W, X, Y such that $W \in X$ and $X \in Y$ but $W \notin Y$.

16. Write the next three rows for the Pascal triangle shown in Fig. 3.4

17. Complete the proof of Theorem 3.1.

18. For sets $A, B, C \subseteq \mathscr{U}$, prove or disprove (with a counterexample), the following: If $A \subseteq B$, $B \not\subseteq C$, then $A \not\subseteq C$.

19. In part (i) of Fig. 3.5 we have the first six rows of Pascal's triangle, where a hexagon centered at 4 appears in the last three rows. If we consider the six numbers (around 4) at the vertices of this hexagon, we find that the two alternating triples — namely, 3, 1, 10 and 1, 5, 6 — satisfy $3 \cdot 1 \cdot 10 = 30 = 1 \cdot 5 \cdot 6$. Part (ii) of the figure contains rows 4 through 7 of Pascal's triangle. Here we find a hexagon centered at 10, and the alternating triples at the vertices — in this case, 4, 10, 15 and 6, 20, 5 — satisfy $4 \cdot 10 \cdot 15 = 600 = 6 \cdot 20 \cdot 5$.

a) Conjecture the general result suggested by these two examples.

b) Verify the conjecture in part (a).

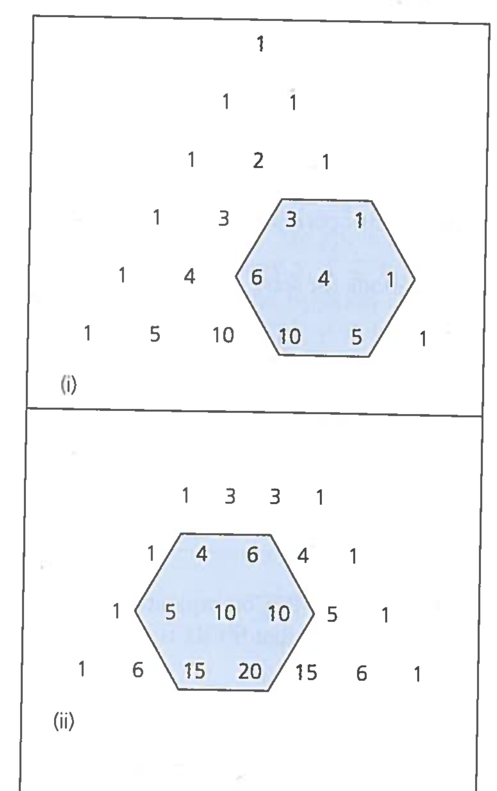

Figure 3.5

20. a) Among the strictly increasing sequences of integers that start with 1 and end with 7 are:

i) 1, 7 ii) 1, 3, 4, 7 iii) 1, 2, 4, 5, 6, 7

How many such strictly increasing sequences of integers start with 1 and end with 7?

b) How many strictly increasing sequences of integers start with 3 and end with 9?

c) How many strictly increasing sequences of integers start with 1 and end with 37? How many start with 62 and end with 98?

d) Generalize the results in parts (a) through (c).

21. One quarter of the five-element subsets of $\{1, 2, 3, \ldots, n\}$ contain the element 7. Determine n (≥ 5).

22. For a given universe $\mathscr{U}$, let $A \subseteq \mathscr{U}$ where A is finite with $|\mathscr{P}(A)| = n$. If $B \subseteq \mathscr{U}$, how many subsets does B have, if (a) $B = A \cup \{x\}$, where $x \in \mathscr{U} - A$? (b) $B = A \cup \{x, y\}$, where $x, y \in \mathscr{U} - A$? (c) $B = A \cup \{x_1, x_2, \ldots, x_k\}$, where $x_1, x_2, \ldots, x_k \in \mathscr{U} - A$?

23. Determine which row of Pascal's triangle contains three consecutive entries that are in the ratio $1:2:3$.

24. Use the recursive technique of Example 3.9 to develop a Gray code for the 16 binary strings of length 4. Then list each of the 16 subsets of the ordered set $\{w, x, y, z\}$ next to its corresponding binary string.

25. Suppose that A contains the elements v, w, x, y, z and no others. If a given Gray code for the 32 subsets of A encodes the ordered set $\{v, w\}$ as 01100 and the ordered set $\{x, y\}$ as 10001, write A as the corresponding ordered set.

26. For positive integers n, r show that

$$\binom{n+r+1}{r} = \binom{n+r}{r} + \binom{n+r-1}{r-1} + \cdots + \binom{n+2}{2} + \binom{n+1}{1} + \binom{n}{0}$$
$$= \binom{n+r}{n} + \binom{n+r-1}{n} + \cdots + \binom{n+2}{n} + \binom{n+1}{n} + \binom{n}{n}.$$

27. In the original abstract set theory formulated by Georg Cantor (1845–1918), a set was defined as "any collection into a whole of definite and separate objects of our intuition or our thought." Unfortunately, in 1901, this definition led Bertrand Russell (1872–1970) to the discovery of a contradiction — a result now known as *Russell's paradox* — and this struck at the very heart of the theory of sets. (But since then several ways have been found to define the basic ideas of set theory so that this contradiction no longer comes about.)

Russell's paradox arises when we concern ourselves with whether a set can be an element of itself. For example, the set