More About Derivatives

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1.6 Differentiation Techniques II: The Product and Quotient Rules

In the last lecture, we saw the definition of the derivative for the first time, and we saw some techniques for computing them efficiently. However, close examination of Theorem 1 from the previous lecture will show you that it really only tells you how to compute for a very restricted set of functions. In particular, we can really only compute derivatives for polynomials, sums of polynomials (which are also polynomials, so we haven't gained anything), some very limited types of rational functions, and sums of these. We will be frequently interested in computing more complicated things. For now, however, we want to take the derivatives we know, and examine what happens if we combine them in products (multiplication) and quotients (division). The situation becomes a little more complicated here, and so we will consider each case individually.

For products of functions, we have the following theorem.

Theorem 1. *(The Product Rule) Suppose $f(x)$ and $g(x)$ are differentiable functions. Then the derivative of the product $f(x)g(x)$ is given by*

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Go back and re-read the previous equation very carefully. This equation is a source of many mistakes on exams. To correctly apply it, you must differentiate only one function at a time, then combine them in the sum.

Example 1. If $f(x) = x^2$, and $g(x) = 2x+1$, find the derivative of $f(x)g(x)$.

Solution: Since we want the derivative of the product of two functions, we apply the formula in Theorem 1. Thus, you should obtain

$$\begin{aligned}
(f(x)g(x))' &= \underbrace{(2x)}_{f'(x)}\underbrace{(2x+1)}_{g(x)} + \underbrace{(x^2)}_{f(x)}\underbrace{(2)}_{g'(x)} \\
&= 4x^2 + 2x + 2x^2 \\
&= 6x^2 + 2x.
\end{aligned}$$

We see here that things are already becoming increasingly more complicated. This is even more so when you consider derivatives for quotients of functions.

Theorem 2. *(The Quotient Rule) Suppose $f(x)$ and $g(x)$ are differentiable functions. Then the derivative of the quotient $f(x)/g(x)$ is given by*

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

There are several tricks to help you remember this formula. One popular mnemonic device is

$$\left(\frac{High}{Low}\right)' = \frac{Low \cdot DHigh - High \cdot DLow}{(Low)^2}.$$

Here, the D just stands for derivative.

Example 2. If $f(x) = x^2$, and $g(x) = 2x+1$, find the derivative of $f(x)/g(x)$.

Solution: Since we want the derivative of the quotient of two functions, we apply the formula in Theorem 2. Thus, you should obtain

$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \frac{\overbrace{(2x)}^{f'(x)}\overbrace{(2x+1)}^{g(x)} - \overbrace{(x^2)}^{f(x)}\overbrace{(2)}^{g'(x)}}{(\underbrace{2x+1}_{g(x)})^2} \\
&= \frac{4x^2 + 2x - 2x^2}{(2x+1)^2} \\
&= \frac{2x^2 + 2x}{(2x+1)^2}.
\end{aligned}$$

1.7 Differentiation Techniques III: The Chain Rule

Theorems 1 and 2 above allow us to extend the set of functions for which we can efficiently compute derivatives. Of course, when I say "efficiently", I mean "without using the limit definition". To see how we are still somewhat limited, consider the function $f(x) = \sqrt{x-1}$. As written, we cannot compute its derivative. It is not a polynomial. It is not a rational function. However, it looks suspiciously like a square root. The problem is that the tricks we learned apply only to $\sqrt{x}$, not to the slightly different $\sqrt{x-1}$. To see why, recall that we can write $\sqrt{x} = x^{1/2}$. Thus, it is exactly in the form allowed by the Power Rule. The function $\sqrt{x-1} = (x-1)^{1/2}$ is not. And as I always say, if it is slightly different, than we cannot use the formula we know. So we need a new trick. This is the content of the following theorem.

Theorem 3. *(Extended Power Rule) If $f(x)$ is differentiable, then for any real number n, we have*

$$((f(x))^n)' = n(f(x))^{n-1} f'(x).$$

What this is telling us is that anything that looks somehow "similar" to x^n can be differentiated in the same way as x^n. This comes at a price, though. You must multiply by the derivative of the function inside the parentheses.

Example 3. Find the derivative of $g(x) = (x^2-1)^2$.

Solution: According to the formula, we have

$$\begin{aligned}
\frac{dg}{dx} &= \frac{d}{dx}\big((\underbrace{x^2-1}_{f(x)})^2\big) \\
&= 2(\underbrace{x^2-1}_{f(x)})^{2-1}(\underbrace{2x}_{f'(x)}) \\
&= 4x(x^2-1).
\end{aligned}$$

Example 4. Find the derivative of $h(x) = (x^3-2x^2+2x-1)^3$.

Solution: According to the formula, we have

$$\begin{aligned}
\frac{dh}{dx} &= \frac{d}{dx}\big((\underbrace{x^3-2x^2+2x-1}_{f(x)})^3\big) \\
&= 3(\underbrace{x^3-2x^2+2x-1}_{f(x)})^{3-1}(\underbrace{3x^2-4x+2}_{f'(x)}) \\
&= (9x^2-12x+6)(x^3-2x^2+2x-1)^2.
\end{aligned}$$

As was the case for Theorems 1 and 2, Theorem 3 allows us to efficiently compute an even greater class of functions. However, it can be improved a bit if we introduce *composition of functions*.

Definition 1. Let $f(x)$ and $g(x)$ be functions. Then we define the *composed function* $f \circ g(x)$ as

$$f \circ g(x) = f(g(x)).$$

So, when we compose two functions, we merely substitute x into the second function, then take the result and substitute it into the first function.

Example 5. Find $f \circ g(x)$ for the combinations of functions given below.

1. $f(x) = x^2$, $g(x) = x - 1$.
2. $f(x) = \frac{1}{x}$, $g(x) = x^2 - 1$.

Solution: Using the definition of composition of functions, we have:

1. $f(g(x)) = f(x-1) = (x-1)^2$.
2. $f(g(x)) = f(x^2 - 1) = \frac{1}{x^2-1}$.

Now that we understand compositions of functions, we can state a more general version of Theorem 3.

Theorem 4. *(The Chain Rule) Suppose $f(x)$ and $g(x)$ are differentiable. Then $f \circ g(x)$ is differentiable, and*

$$(f \circ g(x))' = f'(g(x))g'(x).$$

Since we have worked mostly with combinations of power functions (that is, functions of the form x^n), the Chain Rule doesn't let us do anything that we couldn't already do with the Extended Power Rule (Theorem 3). However, as we learn the derivatives of other basic types of functions, the Chain Rule will allow us to differentiate many new types of functions.