Introduction to Derivatives

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1.3 Rates of Change

In this section, which we will only briefly discuss, we will finally introduce what we mean by "change". Consider the following situation:

> The school year is starting, and you have to drive back to Trondheim from your hometown. Let's pretend that the distance is about 300 km. Since you're not the only person going back to school (or going anywhere, for that matter), you encounter a lot of traffic on the way. So much traffic, in fact, that it takes you about 5 hours to drive to Trondheim. How fast were you driving?

I'm sure we all remember that velocity is nothing but the total distance travelled divided by the total time. Thus, in our particular case, we have

$$v = \frac{\text{distance}}{\text{time}} = \frac{300 \text{ km}}{5 \text{ hr}} = 60 \text{ km/hr}.$$

Which is a nice, slow speed. Slow and steady wins the race, as they say.

But now let's be realistic. It is very unlikely that you were driving at that speed the whole time. If you were to look at the speedometer in your car at any moment while driving, chances are that you would see any number other than 60. So what exactly is the speedometer measuring, then? And what exactly is the meaning of the velocity we computed above?

To answer these questions, consider the problem of finding the average grades on an exam in a course. If you wanted to compute the average grade, you simply add up all the grades, and divide by the number of students in the course. This is very similar to the computation we did above: we added up the total distance, and divided by the number of moments (which we call the time). Thus, the computation tells us an *average* of your speed.

On the other hand, your speedometer is doing something much more complicated. Just as we did in our computation, the speedometer is taking an average velocity. However, it is taking this average over a very, very short period in time. Roughly speaking, lets say that the distance you have travelled at time t is represented by $d(t)$. If we start our average at time t, then we can represent the end of short time interval as $t+h$, where h is a very small number. Using the same formula used above for the velocity, the average velocity over this short time period is

$$v = \frac{d(t+h) - d(t)}{(t+h) - t} = \frac{d(t+h) - d(t)}{h}.$$

If we wanted to let the time interval be as short as possible, we can consider this expression for smaller and smaller values of h. But in the previous lectures, we learned a convenient way of doing this: limits. In the language of limits, we want compute

$$\lim_{h \to 0} \frac{d(t+h) - d(t)}{h}.$$

This is the information that your car's speedometer tells you.

1.4 Derivatives

This is nice and all, but how do we know anything we just said above makes any sense? In our previous discussions, we saw that limits may not even exist. To get around this problem, we use a trick frequently used by mathematicians: we look only at cases where this works, and ignore the ones that don't. This leads us to the following definition.

Definition 1. Let $f(x)$ be a function, and let x be in its domain. The limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

if it exists, is known as the *derivative* of $f(x)$. If the derivative of f exists at a point x, we say that f is *differentiable* at x.

There are several notations used for denoting the derivative of a function. The two most common ways are $f'(x)$, which is read as "f prime of x", and

$$\frac{df}{dx},$$

which is simply read as "d f d x". In addition, physicists will sometimes use the notation $\dot{f}$, which is read as "f dot", to denote derivatives. We will only use the first two of these in this course.

Example 1. Consider the line given by the function $y = 2x + 1$. Find the derivative y'.

Solution: Using the definition, we have

$$\begin{aligned} y' &= \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} \\ &= \lim_{h\to 0} \frac{2(x+h) - 2x}{h} \\ &= \lim_{h\to 0} \frac{2h}{h} \\ &= \lim_{h\to 0} 2 \\ &= 2. \end{aligned}$$

Thus, the derivative is $y' = 2$. Note that this is the same as the slope of the line (the number in front of the x in the equation for the line). This is no coincidence. More details about that later.

Example 2. Let $f(x) = x^2$. Find the derivative $f'(x)$.

Solution: Using the definition, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} 2x + h \\ &= 2x. \end{aligned}$$

1.5 Differentiation Techniques I

As we saw above, computing derivatives can be tedious. Even using shortcuts for computing limits, each derivative may still take several steps. If we had to do this each time we wanted to find a derivative, then society would probably come to a screeching halt within minutes. If you don't believe me, try finding the derivative of the function $f(x) = 10x^{50} - 49x^{10} + 21$ and see how quickly you lose interest.

What we need here are efficient ways to compute derivatives. Luckily for us, there are several shortcuts available, which are contained in the following theorem.

Theorem 1. *Derivatives satisfy the following properties:*

1. *(The Power Rule) If $f(x) = x^n$, then $f'(x) = nx^{n-1}$, for any real number n.*
2. *(Constant Functions) If $f(x) = c$, where c is a fixed number, then $f'(x) = 0$.*
3. *(Constants) If $f(x)$ is differentiable and c is a fixed number, then $(cf(x))' = cf'(x)$.*
4. *(Sum-Difference Rule) If $f(x)$ and $g(x)$ are differentiable, then $(f(x) + g(x))' = f'(x) + g'(x)$.*

Going back to my previous statement, what if we do actually need to compute the derivative of $f(x) = 10x^{50} - 49x^{10} + 21$? Using the shortcuts in Theorem 1, we need only the following short(er) computation.

$$\begin{aligned} f'(x) &= (10x^{50} - 49x^{10} + 21)' \\ &= 10(x^{50})' - 49(x^{10})' + (21)' \\ &= 10(50x^{49}) - 49(10x^9) + 0 \\ &= 500x^{49} - 490x^9. \end{aligned}$$

Example 3. If $f(x) = x^{4.2}$, find $f'(x)$.

Solution: By property 1 of derivatives, we have

$$f'(x) = 4.2x^{4.2-1} = 4.2x^{3.2}.$$

Example 4. If $f(x)$ is given by

$$f(x) = \frac{1}{x}$$

find $f'(x)$.

Solution: As written, $f'(x)$ cannot be computed using any of the above shortcuts. However, recall that

$$\frac{1}{x} = x^{-1}.$$

We can then use property 1 to obtain

$$f'(x) = (-1)x^{-1-1} = -1x^{-2} = -x^{-2} = -\frac{1}{x^2}.$$