

# Systems of Linear Equations

November 24, 2014

## 1.1 Systems of Linear Equations

We will now begin the second topic to be discussed in this course, which is linear algebra. Very briefly, linear algebra is the study of linear systems in an abstract way, in which we determine what general properties such systems have and what we can do with this general structure. Because this course is not meant to be a very technical course, we will not look at things on a very high level, but rather just introduce some of the framework used in the abstract study, and see what kinds of computations we can do with it and what kinds of applications there are to this material.

A good place to start our discussion is to introduce what we mean when we say a linear system. In this course, when we say linear system, we mean a system of linear equations. The simplest example of a linear equation is

$$2x = 1.$$

The thing that makes this a linear equation is that the variable  $x$  is raised to the first power. A linear equation can have more than one variable. For example,

$$y = 3x - 1$$

is a perfectly good example of a linear equation. In fact, you can have as many variables as you like. Thus, the equation

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_{n-1}x_{n-1} + a_nx_n = b$$

is also a linear equation (assuming that each  $a_i$  is a number, and each  $x_i$  is a variable). The important thing is that a linear equation contains sums and differences of only products of one variable and a number.

While studying single equations is ok, things become much more interesting when you consider several equations simultaneously. Such a collection of linear equations is called a *system of linear equations* (which we usually shorten to just *system of equations*, or just *system*).

**Example 1.** The collection of equations given by

$$\begin{aligned}x_1 + x_2 &= 2 \\x_1 - x_2 &= 0\end{aligned}$$

is a system of linear equations.

Note that there should be at least some overlap in the variables contained in the equation, or else there is no reason to study the equations together.

With any equation, no matter what type, the fundamental question is: do we have solutions? Before we can try to answer that question, we must be specific about what we mean with the word “solution”.

**Definition 1.** Given a system of equations containing variables  $x_1, x_2, x_3, \dots, x_{n-1}, x_n$ , a *solution set* is a collection of numbers  $(s_1, s_2, s_3, \dots, s_{n-1}, s_n)$  which, when substituted into each of the equations, will satisfy all equations simultaneously.

**Example 2.** The system

$$\begin{aligned}x_1 + x_2 &= 2 \\x_1 - x_2 &= 0\end{aligned}$$

has  $(1, 1)$  as a solution set.

It is probably obvious that solution sets are not tied to any one system of equations. For example, the solution set  $(1, 1)$  solves not only the above system, but also solves the much simpler system given by

$$\begin{aligned}x_1 &= 1 \\x_2 &= 1.\end{aligned}$$

This observation may seem silly or trivial, but it is, in fact, quite important. As always, when something is important, we give it a name.

**Definition 2.** If two systems of equations have the same solution set, we say they are *equivalent*.

Based on this definition, it follows that the systems

$$\begin{array}{rcl} x_1 + x_2 = 2 & & x_1 = 1 \\ & \text{and} & \\ x_1 - x_2 = 0 & & x_2 = 1 \end{array}$$

are equivalent. However, we can also make another observation. In the first system, suppose we add the equations together. The result is given by the following computation:

$$\begin{array}{r} x_1 + x_2 = 2 \\ + \quad x_1 - x_2 = 0 \\ \hline 2x_1 = 2 \end{array}$$

Now, suppose we subtract the two equations from each other. This results in the following computation:

$$\begin{array}{r} x_1 + x_2 = 2 \\ - \quad x_1 - x_2 = 0 \\ \hline 2x_2 = 2 \end{array}$$

Dividing both of these resulting equations by 2, we obtain the “trivial system”

$$\begin{array}{l} x_1 = 1 \\ x_2 = 1 \end{array}$$

once again. Thus, we took two different systems, which we said were equivalent (because they had the same solutions), and we combined the equations in one system in such a way as to turn one into the other. The interesting thing is that this can always be done, and in fact can be done by using only the following three operations on the equations:

1. Multiplying or dividing an equation by a non-zero number.
2. Adding or subtracting two equations from each other.
3. Switching two equations.

Thus, given any system of equations, we will solve it by attempting to use these three operations to obtain an equivalent system which is “trivial” in the same sense as in the above example.

## Solving Linear Systems

In the previous section, we discussed how we can solve a system of equations by performing certain operations on the equations. The problem is that it may not always be clear how to proceed in solving a given system of equations. Thus, we would like to outline a systematic procedure which can be used to try to find the solutions of the system. This procedure is particularly well suited for programming into a computer, which can then perform the admittedly tedious computations necessary very quickly.

To illustrate this procedure, consider the following system of equations:

$$\begin{array}{r} x_1 + 2x_2 - x_3 = 2 \\ x_1 - x_2 + x_3 = 2 \\ x_2 + x_3 = 5 \end{array}$$

**Step 1:** If the first equation has the first variable (in this case,  $x_1$ ), leave the first equation unchanged in the system. Then use it to get rid of the variable  $x_1$  from the remaining equations by using the three operations listed above. If it does not have  $x_1$ , switch equations so that the first equation has  $x_1$ , then proceed as before.

In this example, only the second equations must have  $x_1$  removed from it. As before, we can subtract the first equation from the second to obtain

$$\begin{array}{r} x_1 + 2x_2 - x_3 = 2 \\ - \quad x_1 - x_2 + x_3 = 2 \\ \hline 3x_2 - 2x_3 = 0 \end{array}$$

What we have obtained here will now become our new second equation. Thus, we now have the (equivalent) system

$$\begin{array}{r} x_1 + 2x_2 - x_3 = 2 \\ 3x_2 - 2x_3 = 0 \\ x_2 + x_3 = 5 \end{array}$$

**Step 2:** If the second equation has the second variable (in this case,  $x_2$ ), leave it unchanged in the system. Then use it to get rid of the variable  $x_2$  from the remaining equations by using the three operations listed above. If it does not have  $x_2$ , switch equations so that the second equation has  $x_2$ , then proceed as before.

In the current example, we only have one equation remaining, which happens to contain  $x_2$  in it. Since our second equation also has  $x_2$ , we can use it to remove the  $x_2$  from the third. If we first multiply the last equation by 3 and subtract, we get the following computation:

$$\begin{array}{r}
 0x_1 + 3x_2 - 2x_3 = 0 \\
 - 0x_1 + 3x_2 + 3x_3 = 15 \\
 \hline
 \phantom{-} \phantom{0x_1 +} - 5x_3 = -15
 \end{array}$$

Dividing the result by  $-5$ , we get that the new equation 3 is  $x_3 = 3$ . Thus the new system is

$$\begin{array}{r}
 x_1 + 2x_2 - x_3 = 2 \\
 3x_2 - 2x_3 = 0 \\
 x_3 = 3
 \end{array}$$

**Step 3:** Continue in this manner until you reach the final equation, which should not contain any of the variables you already eliminated.

Examining the result from the previous step, we see that we have reached the final equation, and it does not contain any  $x_1$  or  $x_2$ .

**Step 4:** If the final equation contains only the final variable, use this equation to remove the final variable (in this case,  $x_3$ ) from the previous equation. Otherwise, you are done.

If we substitute  $x_3 = 3$  into the second equation, we obtain  $3x_2 = 6$ , or  $x_2 = 2$ .

**Step 5:** Repeat Step 4 for each equation, working backwards until you either run out of equations or variables to substitute into.

Now that we know  $x_2$  and  $x_3$ , we substitute into the first equation and get  $x_1 + 4 - 3 = 2$ , or  $x_1 = 1$ . Since we have now solved for each variable, we are done.