Improper Integrals

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5.3 Improper Integrals

Previously, we discussed how integrals correspond to areas. More specifically, we said that for a function $f(x)$, the region created by the graph of f, the x-axis, $x = a$ and $x = b$ will have an area A that is given by

$$A = \int_a^b f(x)\ dx.$$

An interesting question to think about is the following: assuming that $f(x)$ is defined for all real numbers, what happens if we let, say, the upper limit of integration b get larger and larger? For example, suppose we let $f(x) = x$, and $a = 0$. Then

$$A = \int_0^b x\ dx = \frac{1}{2}b^2.$$

We can choose some increasing values for b to determine what happens as b gets larger:

b	A
10	50
100	5000
1000	500000

By now, you can surely tell that as b gets larger, so does the area. In fact, by just choosing b large enough, we can make A as large as we want. In mathematical notation, we write

$$\lim_{b\to\infty} \int_0^b x\ dx = \lim_{b\to\infty} \frac{1}{2}b^2 = +\infty.$$

To simplify notation, we define

$$\int_a^\infty f(x)\ dx = \lim_{b\to\infty} \int_a^b f(x)\ dx.$$

The previous computation should come as no surprise. After all, if you have an infinitely large shape, its area should obviously be infinitely large. As obvious as it may seem, this observation may, in fact, be false.

Consider now the function $f(x) = 1/x^2$, and let us evaluate the integral

$$\int_1^\infty \frac{1}{x^2}\ dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\ dx$$

By using the Fundamental Theorem of Calculus, we compute

$$\begin{aligned}\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\ dx &= \lim_{b\to\infty} -\frac{1}{x}\Big|_0^b \\ &= \lim_{b\to\infty} 1 - \frac{1}{b} \\ &= 1 - \lim_{b\to\infty} \frac{1}{b}.\end{aligned}$$

The funny thing here is that the limit at the end is equal to zero. Thus, we have

$$\int_1^\infty \frac{1}{x^2}\ dx = 1.$$

This is an odd result. It says that we can have an infinitely long shape which has a finite area! This is one of the many counterintuitive results in mathematics which makes the subject interesting. This leads us to the following definitions.

Definition 1. An *improper integral* is an integral of the form

$$\int_a^\infty f(x)\ dx \quad \text{or} \quad \int_\infty^b f(x)\ dx.$$

To compute such an integral, we use the definitions

$$\int_a^\infty f(x)\ dx = \lim_{b\to\infty} \int_a^b f(x)\ dx$$
$$\int_{-\infty}^b f(x)\ dx = \lim_{a\to-\infty} \int_a^b f(x)\ dx$$

If the limits exist and are finite, we say the integral *converges*. Otherwise, we say the integral *diverges*.

Example 1. Determine if the integral

$$\int_{-\infty}^{0} e^x \; dx$$

converges or diverges.

Solution: We start by applying the definition:

$$\int_{-\infty}^{0} e^x \; dx = \lim_{a \to -\infty} \int_{a}^{0} e^x \; dx$$

Next, we apply the Fundamental Theorem of Calculus:

$$\begin{aligned} \lim_{a \to -\infty} \int_{a}^{0} e^x \; dx &= \lim_{a \to -\infty} e^x \Big|_a^0 \\ &= \lim_{a \to -\infty} e^0 - e^a \\ &= \lim_{a \to -\infty} 1 - e^a \end{aligned}$$

Finally, we recall the fact (which can be obtained by examining the graph of e^x):

$$\lim_{a \to -\infty} e^a = 0.$$

So we obtain

$$\int_{-\infty}^{0} e^x \; dx = 1.$$

Since this is a finite number, the integral converges, and we say that it converges to 1.

Example 2. Determine if the integral

$$\int_{1}^{\infty} \frac{1}{x} \; dx$$

converges or diverges.

Solution: We proceed as before, by first writing the definition:

$$\int_1^{\infty} \frac{1}{x}\, dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x}\, dx$$

Applying the Fundamental Theorem of Calculus, we get

$$\begin{aligned}\lim_{b\to\infty} \int_1^{b} \frac{1}{x}\, dx &= \lim_{b\to\infty} \ln b - \ln 1 \\ &= \lim_{b\to\infty} \ln b.\end{aligned}$$

Recalling the fact (which you can also get by looking at the graph of $\ln x$)

$$\lim_{b\to\infty} \ln b = +\infty,$$

we thus get that

$$\int_1^{\infty} \frac{1}{x}\, dx = +\infty.$$

So the integral diverges.

5.7 Differential Equations

A concept which has been of crucial importance in modern physics is that of a *differential equation*. Roughly speaking, a differential equation is simply an equation involving an unknown function y and any combination of its derivatives. The goal is to then find a function $y = f(x)$ which satisfies the equation.

Example 3. A classic example of a differential equation is Newton's Second Law of Motion, which states that the force applied to an object is equal to the product of its mass and acceleration. Recalling that the acceleration a is the second derivative of position s, this gives us $F = ms''$.

Consider now a spring with one end attached to a fixed object. If we let s be the distance of the other end from its rest position, then Hooke's Law tells us that the force on the spring is given by $F = ks$, where k is an appropriate constant. Substituting this into Newton's Second Law gives us $ks = ms''$. This equation appears in many branches of science, particularly physics, chemistry, and engineering. It has solutions which look like

$$s(t) = A\cos(\omega t) + B\sin(\omega t),$$

where A and B are just constants, and

$$\omega = \sqrt{\frac{k}{m}}$$

is called the *frequency*. If we recall that the trigonometric functions $\sin t$ and $\cos t$ oscillate back and forth repeatedly between -1 and 1 as t increases, we can see why any system which obeys this equation is called a *harmonic oscillator*.

While differential equations are an extremely important tool for modern science, it is a very annoying subject in that there is not one method of solving them which works for all types of equations. In fact, for most equations, there is no known method of solving them (though we often know that solutions exist somehow). In addition, the methods for solving equations can be quite complicated. Because of this, we will focus our attention on only the two most simple types of differential equations which can be solved explicitly, explained in the following definition.

Definition 2. Let y be an unknown function. A Type I differential equation is an equation of the form

$$\frac{dy}{dx} = f(x),$$

where $f(x)$ is a given function. A Type II differential equation is an equation of the form

$$\frac{dy}{dx} = f(x)g(y),$$

where $f(x)$ and $g(y)$ are given functions.

Solving Type I Equations

To solve Type I equations, we begin with their defining equation:

$$\frac{dy}{dx} = f(x)$$

Integrating both sides gives

$$\int \frac{dy}{dx}\ dx = \int f(x)\ dx.$$

On the left side, we recall that $\frac{dy}{dx}\ dx = dy$. Thus

$$\int dy = \int f(x)\ dx.$$

If we integrate the left side with respect to y, we simply obtain $y + C$. However, since the integral on the right-hand side will also add another constant C, we simply combine the two constants. Thus, we may simply write

$$y = \int f(x)\ dx.$$

Example 4. Solve the differential equation

$$\frac{dy}{dx} = 2x.$$

Solution: Based on the previous computation, y is given by

$$y = \int 2x\ dx = \frac{1}{2}x^2 + C.$$

Example 5. Solve the differential equation

$$\frac{dy}{dx} = \cos(x)e^{\sin(x)}$$

Solution: Based on the previous computation, y is given by

$$y = \int \cos(x)e^{\sin(x)}\ dx = e^{\sin(x)} + C.$$

Solving Type II Equations

As in the case of Type I equations, we try to solve these equations by simply integrating. Recall that Type II equations look like

$$\frac{dy}{dx} = f(x)g(y).$$

The difficulty is that now the right hand side may contain expressions involving y. We try to get around this by removing anything with y by rewriting it as

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x).$$

We then try integrating both sides:

$$\int \frac{1}{g(y)} \frac{dy}{dx}\ dx = \int f(x)\ dx.$$

The right hand side looks like something we can handle. The left hand side, however, looks a little scary. However, we can make it look a little nicer if we use differentials. Recall that

$$\frac{dy}{dx}\ dx = dy.$$

Then we can rewrite

$$\int \frac{1}{g(y)} \frac{dy}{dx}\ dx = \int \frac{1}{g(y)}\ dy.$$

Thus

$$\int \frac{1}{g(y)}\ dy = \int f(x)\ dx.$$

It may look like we haven't improved the left side at all, but depending on what $g(y)$ looks like, the integral may be quite simple. Suppose that

$$\int \frac{1}{g(y)}\ dy = G(y)$$

for some function $G(y)$. Then we can at least find an implicit solution in the form

$$G(y) = \int f(x)\ dx.$$

If this is something you can solve for y explicitly, then we have found our solution.

Example 6. Solve the differential equation

$$\frac{dy}{dx} = xy.$$

Solution: We start by moving the y to the left side:

$$\frac{1}{y} \frac{dy}{dx} = x$$

Next, we integrate

$$\int \frac{1}{y}\frac{dy}{dx}\ dx = \int x\ dx$$

We rewrite the left side as

$$\int \frac{1}{y}\frac{dy}{dx}\ dx = \int \frac{1}{y}\ dy = \ln|y| + C,$$

and so we get

$$\ln|y| = \int x\ dx = \frac{1}{2}x^2 + C.$$

Thus we can get at least an implicit solution to the equation. We can, however, go a little further by doing the following: Suppose we want to cancel the ln from the left side. We can use the following trick: recall that the functions e and ln cancel each other out. Thus

$$e^{\ln|y|} = |y|.$$

This tells us that

$$\begin{aligned} |y| &= e^{\frac{1}{2}x^2+C} \\ &= Ce^{\frac{1}{2}x^2}. \end{aligned}$$

To get rid of the absolute value, we note that $e^{\frac{1}{2}x^2}$ is always positive. Since y is a product of $e^{\frac{1}{2}x^2}$ and C, we note that it only matters whether C is positive or negative. If $C > 0$, then $|y| = y$, and we are done. If $C < 0$, then $|y| = -y = Ce^{\frac{1}{2}x^2}$. But we can then move the negative to the C:

$$y = -Ce^{\frac{1}{2}x^2},$$

and hide it by renaming C to $-C$ (using the trick of using the same name for everything). Thus

$$y = Ce^{\frac{1}{2}x^2}.$$

Initial Value Problems

In many applications of differential equations, we need to find a solution to the equation which satisfies some condition. Frequently, these conditions consist of knowing the value of the unknown function y at a specified x-value. Such types of problems are known as *initial value problems*, because we are given $y = y_0$ for some $x = x_0$, which often represents the time at which the system starts. To solve such problems, we proceed as before, applying the initial condition $y(x_0) = y_0$ at the very end. This will force the constant C to be a specific number, as only one solution will be able to satisfy the initial condition. Because of this, we refer to solutions of initial value problems as *specific solutions*.

Example 7. Solve the initial value problem

$$\frac{dy}{dx} = y$$
$$y(0) = -2.$$

Solution: We proceed as before, separating x's and y's:

$$\frac{1}{y}\frac{dy}{dx} = 1$$

We then integrate, which (after applying the definition of differentials), gives us

$$\int \frac{1}{y}\,dy = dx.$$

Evaluating the integral gives

$$\ln|y| = x + C$$

Solving for y in the same way as in the previous example gives us

$$|y| = Ce^x$$

To continue, we must be very careful here. We know that $y(0) = -2$. You may be tempted to just say

$$|-2| = Ce^0 = C,$$

and so you may think that $C = 2$. However, the function $y(x) = 2e^x$ does not satisfy the condition $y(0) = -2$. As we said before, when your initial condition is negative, the constant C must be negative. Thus C is *really* $C = -2$. Thus, the specific solution is

$$y(x) = -2e^x.$$