Linear Algebra

Linear Systems

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Linear Systems

We already know how to solve a 2×2 linear system of the form

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2.$$

Linear Systems

Example 1

$$\begin{aligned} 5x + 2y &= 90 \\ 3x + 4y &= 96. \end{aligned}$$

The first equation divided by 5 yields the system

$$\begin{aligned} x + \frac{2}{5}y &= 18 \quad (\Longleftrightarrow\ 3x + \frac{6}{5}y = 54) \\ 3x + 4y &= 96. \end{aligned}$$

The new form of the second equation is obtained by subtracting the first equation multiplied by 3 from the second equation:

$$\begin{aligned} x + \frac{2}{5}y &= 18 \\ \frac{14}{5}y &= 42. \end{aligned}$$

This gives us $y = 15$ and hence $x = 12$. So, the system has a unique solution $(x, y) = (12, 15)$.

Linear Systems

Example 2

$$2x + y = 1$$
$$2x + y = 2.$$

This system has no solutions.

Linear Systems - Graphical Interpretation

Solving a linear 2×2 system has a graphical interpretation. Every equation of the form

$$ax + by = c \quad (a \neq 0 \text{ or } b \neq 0)$$

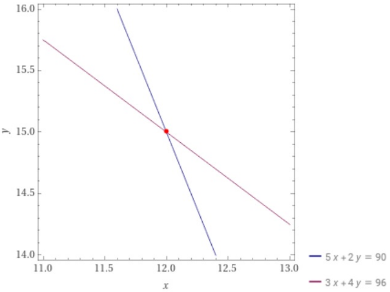
represents a line in the plane. Note that if $b \neq 0$, we have $ax + by = c \iff y = -\frac{a}{b}x + \frac{c}{b}$. When we solve the system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2, \end{aligned}$$

we find the points of intersection of two lines.

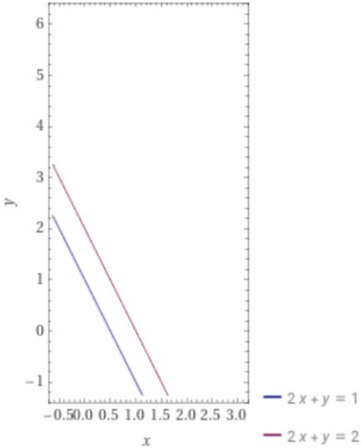
Linear Systems - Graphical Interpretation

When the lines have a unique point of intersection, the system has a unique solution, see Example 1:



Linear Systems - Graphical Interpretation

When the two lines are parallel, the system has no solution, see Example 2:

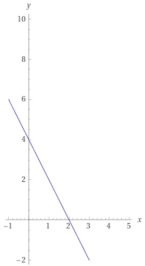


Linear Systems - Graphical Interpretation

The two lines coincide precisely when the system has infinitely many solutions (i.e., when we actually have one equation instead of two). For example, consider the system

$$2x + y = 4$$
$$4x + 2y = 8.$$

Both equations yield $y = -2x + 4$.



Linear Systems

How do we find the infinitely many solutions?

We introduce a new parameter t. We set one of the variables of the system, for example x, to be equal to t:

$$x = t, \quad t \in \mathbb{R}.$$

We then use the equations of the system to express the other variable, here y, in terms of t:

$$2x + y = 4 \Longrightarrow y = 4 - 2x = 4 - 2t.$$

The infinitely many solutions of the system have the form

$$(x, y) = (t, 4 - 2t), \quad t \in \mathbb{R}.$$

What does this mean? For example, when $t = 5$, the pair $(x, y) = (5, -6)$ is a solution. When $t = -1$, the pair $(x, y) = (-1, 6)$ is another solution.

Linear Systems

We could also have set the other variable equal to t: $y = t$. Then $2x + t = 4$ gives $x = 2 - \frac{1}{2}t$ and thus, solutions have the form $\left(2 - \frac{1}{2}t, t\right)$.

Linear Systems

More generally, an $m \times n$ system of linear equations (or $m \times n$ linear system) consists of m equations and n variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m. \end{aligned}$$

A system of the above form can be solved by means of the **Gauss elimination process**.

Gauss Elimination Process

In the Gauss elimination process, we are allowed to perform 3 operations:

1. multiply/divide a row by a scalar $a \neq 0$.
2. add a multiple of a row by $a \neq 0$ to another row.
3. interchange the position of rows.

Gauss Elimination Process

Example 1

Solve

$$3x + 5y - z = 10$$
$$2x - y + 3z = 9$$
$$4x + 2y - 3z = -1.$$

Solution:

We obtain Row 3 by subtracting 2 times Row 2 from Row 3:

$$\begin{aligned} 3x + 5y - z &= 10 \\ 2x - y + 3z &= 9 \\ 4y - 9z &= -19. \end{aligned}$$

Gauss Elimination Process

We obtain Row 1 by multiplying with 2 and we get Row 2 by multiplying with 3:

$$\begin{aligned} 6x + 10y - 2z &= 20 \\ 6x - 3y + 9z &= 27 \\ 4y - 9z &= -19. \end{aligned}$$

Now we can subtract the first row from the second row, and we obtain the new Row 2:

$$\begin{aligned} 6x + 10y - 2z &= 20 \\ -13y + 11z &= 7 \\ 4y - 9z &= -19. \end{aligned}$$

Gauss Elimination Process

To obtain Row 2, we multiply it by 4. Moreover, multiplying Row 3 with 13 yields the new Row 3:

$$\begin{aligned} 6x + 10y - 2z &= 20 \\ -52y + 44z &= 28 \\ 52y - 117z &= -247. \end{aligned}$$

Adding Row 3 and Row 2 gives the new form of Row 3:

$$\begin{aligned} 6x + 10y - 2z &= 20 \\ -52y + 44z &= 28 \\ -73z &= -219. \end{aligned}$$

This gives $z = 3$, $y = 2$ and $x = 1$. Therefore, the system has a unique solution $(x, y, z) = (1, 2, 3)$.

Example 2

Solve

$$\begin{aligned} 2x - y + z &= 3 \\ 4x - 4y + 3z &= 2 \\ 2x - 3y + 2z &= 1. \end{aligned}$$

Solution:

In order to get Row 3, we multiply it by 2 and subtract Row 2:

$$\begin{aligned} 2x - y + z &= 3 \\ 4x - 4y + 3z &= 2 \\ -2y + z &= 0. \end{aligned}$$

Gauss Elimination Process

To obtain Row 2, we subtract 2 times Row 1 from Row 2:

$$\begin{aligned} 2x - y + z &= 3 \\ -2y + z &= -4 \\ -2y + z &= 0. \end{aligned}$$

Hence, the system has no solutions.

Example 3

Solve

$$\begin{aligned} x - 3y + z &= 4 \\ x - 2y + 3z &= 6 \\ 2x - 6y + 2z &= 8. \end{aligned}$$

Solution:

We subtract the first row from the second row to obtain Row 2:

$$\begin{aligned} x - 3y + z &= 4 \\ y + 2z &= 2 \\ 2x - 6y + 2z &= 8. \end{aligned}$$

Gauss Elimination Process

We obtain the last row by subtracting 2 times Row 1 from Row 3:

$$\begin{aligned} x - 3y + z &= 4 \\ y + 2z &= 2 \\ 0z &= 0. \end{aligned}$$

We can see that whatever value z has, the last equation $0 \cdot z = 0$ always holds true. This means that the system has infinitely many solutions (the variable z is "free"). We set $z = t$, $t \in \mathbb{R}$. We get from the second equation

$$y + 2z = 2 \quad \Longrightarrow \quad y + 2t = 2 \quad \Longrightarrow \quad y = 2 - 2t, \quad t \in \mathbb{R}.$$

The first equation then yields

$$x - 3y + z = 4 \quad \Longrightarrow \quad x - 3(2 - 2t) + t = 4$$

$$\Longrightarrow \quad x - 6 + 6t + t = 4 \quad \Longrightarrow \quad x = 10 - 7t, \quad t \in \mathbb{R}.$$

Therefore, the system has infinitely many solutions of the form

$$(x, y, z) = (10 - 7t, 2 - 2t, t), \quad t \in \mathbb{R}.$$