# Differential Equations

# Equilibrium and Stability

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# Autonomous DEs

Here, we will consider autonomous differential equations of the form

$$\boxed{\frac{dy}{dx} = g(y).} \tag{1}$$

If we can solve this DE, then we can study the solution directly to obtain information about its long-term behavior. But what if we want to study the behavior of $y(x)$ as $x \to \infty$, but we cannot solve the DE (1)?

# Point Equilibrium

## Definition: Point Equilibrium

We say that the constant function $y(x) = \hat{y}$ is a **point equilibrium** (or simply **equilibrium**) for the DE (1) if

$$\boxed{g(\hat{y}) = 0.}$$

# Point Equilibrium

## Example

Find the point equilibrium of the DE $y' = 2(y+1)(y-2)$.

**Solution:** This is a DE of the form $\frac{dy}{dx} = g(y)$ with $g(y) = 2(y+1)(y-2)$. In order to find the point equilibria, we solve

$$\begin{aligned} g(y) = 0 \quad &\Longleftrightarrow \quad 2(y+1)(y-2) = 0 \\ &\Longleftrightarrow \quad y = -1 \text{ or } y = 2. \end{aligned}$$

The equilibria are the two functions

$$y_1(x) = -1 \text{ and } y_2(x) = 2.$$

# Point Equilibrium

Why are these functions called equilibria?

If we solve the DE $\frac{dy}{dx} = g(y)$ with initial condition $y = \hat{y}$, where $\hat{y}$ is an equilibrium, then we find the constant solution $y = \hat{y}$.

## Logistic equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

We see from the logistic equation that if $N = K$ or $N = 0$, then $dN/dt = 0$, implying that $N(t)$ is constant, hence, $N = K$ and $N = 0$ are point equilibria. **The solution of a DE can inform us about long-term behavior**, as we see in the case of logistic growth: If the initial condition is $N_0 = 0$, then $N(t) = 0$ for all $t > 0$. Moreover, when $N_0 = K$, then $N(t) = K$ for all $t > 0$.

# Stability of Equilibria

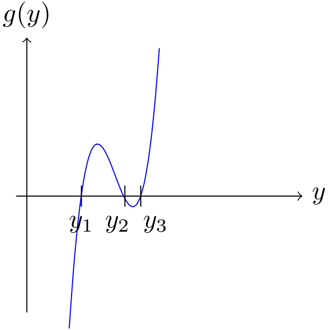
## Definition: Stability

Assume $\hat{y}$ is a point equilibrium of $y' = g(y)$.

- If $g'(\hat{y}) < 0$, then the equilibrium $\hat{y}$ is called **locally stable**.
- If $g'(\hat{y}) > 0$, then the equilibrium $\hat{y}$ is called **unstable**.

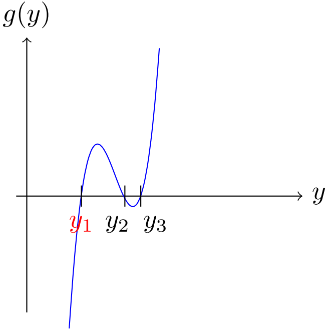
In other words, $\hat{y}$ is locally stable if the solution returns to $\hat{y}$ after a small perturbation; this means that we look at what happens to the solution when we start close to the equilibrium (i.e., the solution moves away from the equilibrium by a small amount, called a small perturbation). If the solution does not return to the equilibrium after a small perturbation, $\hat{y}$ is unstable.

# Stability of Equilibria



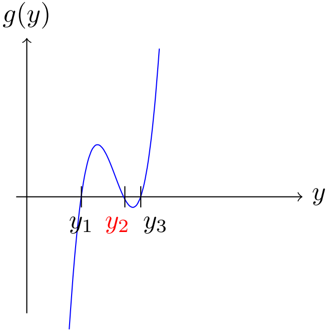

The equilibria are the points of intersection of $g(y)$ with the horizontal axis (solutions of $g(y) = 0$). In this example, the equilibria are $y_1$, $y_2$ and $y_3$.

# Stability of Equilibria




If the function $g(y)$ is increasing near an equilibrium $\hat{y}$, then $\hat{y}$ is unstable. Here, $y = y_1$ is unstable. If we perturb $y_1$ a bit, say $y = y_1 + z$ ($z > 0$ small), then $y'(x) = g(y) > 0$, and so $y$ will increase. If we let $y = y_1 + z$ ($z < 0$ small), then $y'(x) = g(y) < 0$, and so $y$ will decrease. In either case, the system will not return to $y_1$.

# Stability of Equilibria




If $g(y)$ is decreasing near $\hat{y}$, then $\hat{y}$ is stable. For example, if we perturb $y_2$ a bit, say $y = y_2 + z$ ($z > 0$ small), then $y'(x) = g(y) < 0$, and so $y$ will decrease. If we let $y = y_2 + z$ ($z < 0$ small), then $y'(x) = g(y) > 0$, and so $y$ will increase. The system will therefore return to the equilibrium $y_2$ after a small perturbation.

In our example, $y_2$ is locally stable while $y_1$ and $y_3$ are unstable.

# Stability of Equilibria

## Example

Consider the DE

$$\frac{dN}{dt} = y(y+1).$$

What are the equilibria of this DE? Classify them according to stability.

**Solution:** This is a DE of type $\frac{dN}{dt} = g(y)$, with $g(y) = y(y+1)$. Let us find the equilibria:

$$g(y) = y(y+1) = 0 \quad \Longleftrightarrow \quad y = 0 \text{ or } y = -1.$$

Hence, the equilibria are $y_1 = 0$ and $y_2 = -1$.
We have $g(y) = y^2 + y$ and so $g'(y) = 2y + 1$. Hence $g'(0) = 1 > 0$ and thus $y_1 = 0$ is unstable. Moreover, $g'(-1) = -2 + 1 = -1 < 0$, and thus $y_2 = -1$ is locally stable.

# Stability of Equilibria

## Example

Classify the stability of the equilibria of the logistic equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right).$$

Explain what stability means for the polulation of $N(t)$.

**Solution:**

$$\frac{dN}{dt} = g(N), \quad \text{where} \quad g(N) = rN\left(1 - \frac{N}{K}\right).$$

$$g(N) = 0 \quad \Longleftrightarrow \quad N = 0 \text{ or } N = K.$$

$$g'(N) = r\left(1 - \frac{N}{K}\right) - \frac{rN}{K} = r - \frac{2rN}{K}$$

$g'(0) = r > 0 : N = 0$ is an unstable equilibrium

$g'(K) = -r < 0 : N = K$ is a locally stable equilibrium

# Stability of Equilibria

$N = 0$: When we perturb the initial condition $N_0 = 0$ by a small quantity, the population $N(t)$ will not evolve towards the direction of returning to $0$.

$N = K$: On the contrary, when we perturb the initial condition $N_0 = K$ by a small quantity, the population $N(t)$ will try to return to $K$.