Differential Equations

The Logistic Equation

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The Logistic Equation

The Logistic Equation is

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad N(0) = N_0,$$

where $r, K > 0$ are constants. To find the solution mathematically, we use separation of variables and find

$$N(t) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right)\exp(-rt)}.$$

The Logistic Equation

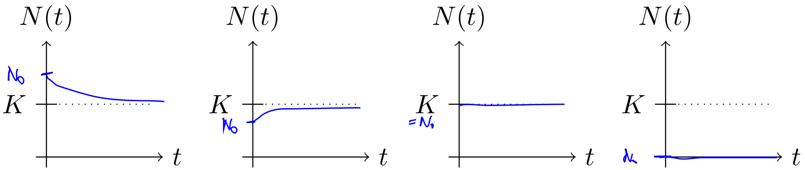
We now proceed to the explanation of the logistic equation as a growth model. We obtain

$$\lim_{t \to +\infty} N(t) = K.$$

K is called the **carrying capacity** of the population.

The Logistic Equation

Graphs of the solution $N(t)$ for different values of N_0:

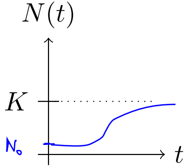


We see that:

- When $N_0 > K$, the population decreases asymptotically towards K.
- When $N_0 < K$, the population increases towards the value K.
- When $N_0 = K$, the population remains constant and equal to K. This corresponds to the constant solution $N(t) = K$ of the DE.
- When $N_0 = 0$, the population remains equal to 0 (there is nothing to reproduce).

The Logistic Equation

Let $0 < N_0 < K/2$:



We can show that if $0 < N_0 < K/2$, then the solution curve is S-shaped. An S-shaped curve is characteristic of populations that show this type of density-dependent growth. At low densities, the growth is almost like unrestricted growth. At higher densities, the growth is restricted and the curve bends around and eventually levels off at the carrying capacity.

The Logistic Equation

What does the logistic equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad N(0) = N_0$$

imply about the population growth? According to the DE, the per capita growth rate

$$\frac{\frac{dN}{dt}}{N} = r\left(1 - \frac{N}{K}\right)$$

is not constant, but depends on the "density" $\frac{N}{K}$. To be more specific, $\frac{\frac{dN}{dt}}{N}$ is proportional to $1 - \frac{N}{K}$. The bigger the "density" is, the smaller the per capita growth rate gets. $\Longrightarrow$ Compare and contrast with the exponential growth rate, where the per capita growth rate is constant.

The Logistic Equation