Lecture 6

We now focus more on DEs and their meaning within the context of applications.

Pure-Time Differential Equations

These are DEs of the form

$$\frac{dy}{dt} = \underline{f(t)}.$$

(The rate of change of $y(t)$ depends solely on the time t).

Example: The volume $V(t)$ of a cell at time t changes according to the DE

$$\frac{dV}{dt} = \sin t \quad , \quad \underbrace{V(0) = 3}_{\text{at time } t=0 \text{, the cell has a volume of } 3.}$$

Find $V(t)$.

$$\frac{dV}{dt} = \sin t \implies dV = \sin t \, dt$$

$$\implies \int 1 \, dV = \int \sin t \, dt$$

$$\implies V = -\cos t + C$$

$$V(0) = 3 \implies V(0) = -\cos 0 + C$$

$$= -1 + C = 3$$

$$\implies C = 4$$

Therefore : $\underline{\underline{V(t) = 4 - \cos t}}$

Autonomous Differential Equations

These are DEs of the form

$$\frac{dy}{dx} = g(y),$$

where the right-hand side does not explicitly depend on x.

Why are they called autonomous?

Consider the growth model

$$\boxed{\frac{dN}{dt} = 2N,}$$

where $N(t)$ is the population as a function of time.

We have $\frac{dN}{dt} = 2N \Rightarrow \int \frac{1}{N}\, dN = \int 2\, dt$

$$\Rightarrow \ln|N| = 2t + c$$

$$\Rightarrow N = e^{2t} \cdot \underset{\sim}{e^{c}}$$

$$\Rightarrow N(t) = \underline{C \cdot e^{2t}} \quad (C \in \mathbb{R}, \text{ constant}).$$

The growth rate $\frac{dN}{dt}$ does <u>not</u> depend on time, it depends only on N!

$\Rightarrow$ DE is "autonomous":

$$t=0: \quad N(0)=20 \quad \Rightarrow N(0) = C\cdot \underbrace{e^{2\cdot 0}}_{=1} = C = 20$$

$$\Rightarrow \underline{\underline{N(t) = 20\,e^{2t}}}$$

$$t=10: \quad N(10) = 20 \quad \Rightarrow N(10) = C\cdot e^{2\cdot 10}$$

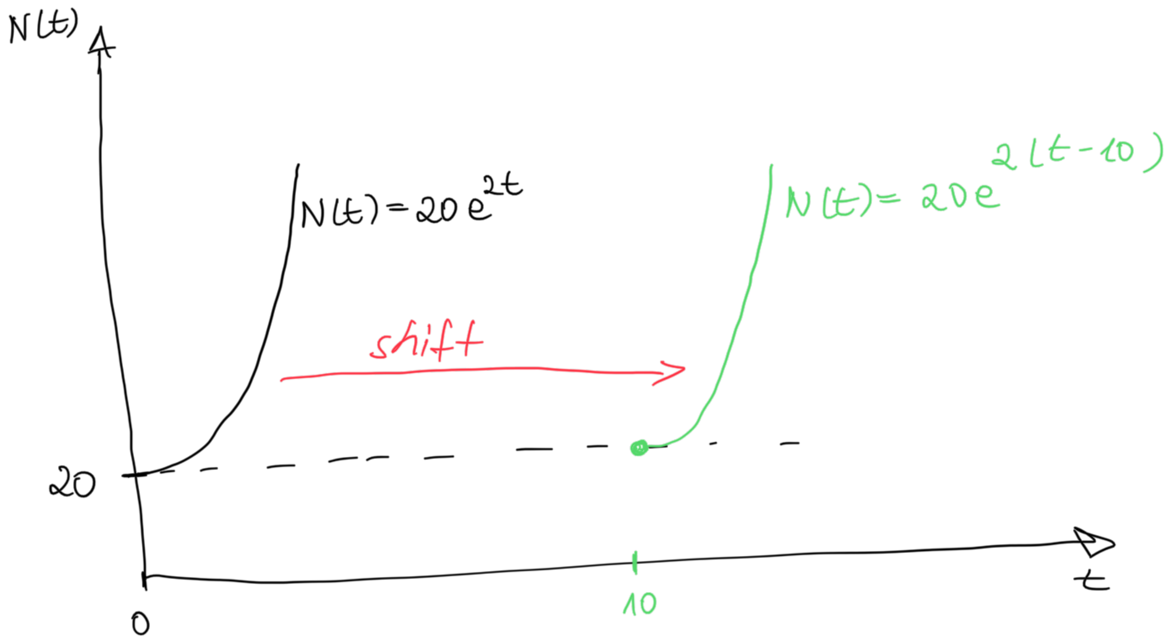
$$= C\cdot e^{20} = 20$$

$$\Rightarrow C = \frac{20}{e^{20}}$$

$$\Rightarrow \underline{\underline{N(t)}} = \frac{20}{e^{20}}\cdot e^{2t}$$

$$= 20\cdot e^{2t-20}$$

$$\underline{\underline{= 20\cdot e^{2(t-10)}}}$$



$\Rightarrow$ Regardless of when we start the ... population follows the experiment, the population ... same trajectory, or, in other words, the growth of the population after the initial moment (here, $t=0$ and $t=10$) is the same.

That was expected, because the growth rate $\frac{dN}{dt}$ is independent of the time t. That is why these DEs are called autonomous.

Examples of Autonomous DEs:

(i) Solve $\frac{dy}{dx} = 2-3y$ with $y(1) = 1$.

Solution: $\frac{dy}{dx} = 2-3y \Rightarrow \frac{dy}{2-3y} = dx$

$$\Rightarrow \int \frac{1}{2-3y}\, dy = \int dx$$

Substitution: $u = 2-3y \Rightarrow \frac{du}{dy} = -3$

$$\Rightarrow dy = -\frac{1}{3}\, du$$

$$\Rightarrow \int \frac{1}{2-3y}\, dy = \int \left(-\frac{1}{3}\right) \cdot \frac{1}{u}\, du$$

$$= -\frac{1}{3} \cdot \ln|u| + c_1$$

$$= -\frac{1}{3} \ln|2-3y| + c_1$$

$$= \int dx = x + c_2$$

$$\Rightarrow \ln|2-3y| = -3x + c$$

$$\Rightarrow |2-3y| = e^{-3x+c} = e^{-3x} \cdot \overline{c}$$

$$\Rightarrow 2-3y = c\,e^{-3x}$$

$$\Rightarrow \underline{\underline{y}} = \frac{c \cdot e^{-3x} - 2}{-3} = -\frac{c}{3} \cdot e^{-3x} + \frac{2}{3}$$

$$= \underline{\underline{c\,e^{-3x} + \frac{2}{3}}}$$

We need to find the value of c:

$$y(1) = 1 \Rightarrow y(1) = c \cdot e^{-3 \cdot 1} + \frac{2}{3} = 1$$

$$\Rightarrow c \cdot e^{-3} = \frac{1}{3}$$

$$\Rightarrow \underline{\underline{c = \frac{1}{3} \cdot e^{3}}}$$

If we substitute the value of c in the solution, we get:

$$\underline{\underline{y}} = \frac{1}{3} e^{3} \cdot e^{-3x} + \frac{2}{3}$$

$$= \underline{\underline{\frac{1}{3} e^{3-3x} + \frac{2}{3}}}$$

(ii) Solve $\frac{dy}{dx} = 2(y-1)(y+2)$

with initial value $y_0 = 2$ when $x_0 = 2$

(i.e., $y(2) = 2$).

Solution:

$$\frac{dy}{dx} = 2(y-1)(y+2)$$

$$\Rightarrow \boxed{\frac{dy}{(y-1)(y+2)}} = 2\,dx$$

We use partial fraction decomposition:

$$\boxed{\frac{1}{(y-1)(y+2)}} = \frac{A}{y-1} + \frac{B}{y+2}$$

$$= \frac{A(y+2) + B(y-1)}{(y-1)(y+2)}$$

$$= \frac{(A+B)y + 2A - B}{(y-1)(y+2)}$$

$$\Rightarrow \underbrace{A + B = 0}_{A = -B} \quad \text{and} \quad \underbrace{2A - B = 1}_{2\cdot(-B) - B = -3B = 1}$$

$$A = -B \longrightarrow 2\cdot(-B) - B = -3B = 1$$

$$\Rightarrow B = -\frac{1}{3}$$

$$A = \frac{1}{3}$$

$$\Rightarrow \int \frac{1}{(y-1)(y+2)}\,dy = \int \left(\frac{1}{3(y-1)} - \frac{1}{3(y+2)} \right) dy$$

$$= \frac{1}{3} \int \left(\frac{1}{y-1} - \frac{1}{y+2} \right) dy$$

$$= \frac{1}{3}\left(\ln|y-1| - \ln|y+2|\right) + c$$

$$= \frac{1}{3} \cdot \ln\left|\frac{y-1}{y+2}\right| + c$$

$$= 2\int dx = 2x + \bar{c}$$

$$\Rightarrow \ln\left|\frac{y-1}{y+2}\right| = 6x + c$$

$$\Rightarrow \frac{y-1}{y+2} = e^{6x} \cdot c$$

We find the value of C:

$$y(2) = 2 \Rightarrow \frac{2-1}{2+2} = \frac{1}{4} = e^{6\cdot 2} \cdot c$$

$$\Rightarrow c = \frac{1}{4} \cdot e^{-12}.$$

If we substitute to the solution of the DE, we get:

$$\frac{y-1}{y+2} = \frac{1}{4} e^{6x} \cdot e^{-12} = \frac{1}{4} \cdot e^{6(x-2)}$$

$$\Rightarrow 4y - 4 = e^{6(x-2)} \cdot (y+2)$$

$$\Rightarrow 4y - ye^{6(x-2)} = 2e^{6(x-2)} + 4$$

$$y\left(4 - e^{6(x-2)}\right) = 2e^{6(x-2)} + 4$$

$$\Rightarrow y = \frac{2e^{6(x-2)} + 4}{4 - e^{6(x-2)}}$$

Growth Models

I. Exponential Population Growth

$$\left.\begin{aligned} \frac{dN}{dt} &= rN \\ N(0) &= N_0 \end{aligned}\right\} \quad (*)$$

$N = N(t)$ ... the population at time t
r ... a paramete

r is called the "intrinsic growth rate".
r is equal to the "per capita growth rate", because

$$r = \frac{\frac{dN}{dt}}{N}$$

when $r > 0$, $(*)$ describes an increasing populatio
When $r < 0$, $(*)$ —"— decreasing —"—
When $r = 0$, the population stays at the same level of the initial condition.

Let us now solve $(*)$:

$$\frac{dN}{dt} = r \cdot N \implies \int \frac{1}{N}\, dN = \int r\, dt$$

$$\implies \ln |N| = rt + c$$

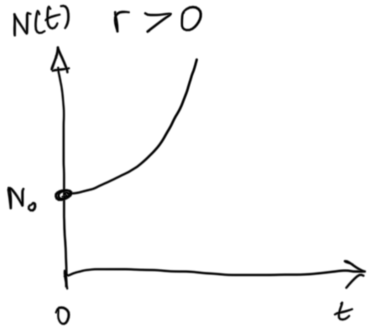
$$\Longrightarrow \quad \underline{\underline{N(t) = e^{rt} \cdot C}} \quad , \quad C \in \mathbb{R}$$

We find C: When $t = 0$, $N(0) = N_0$ and so

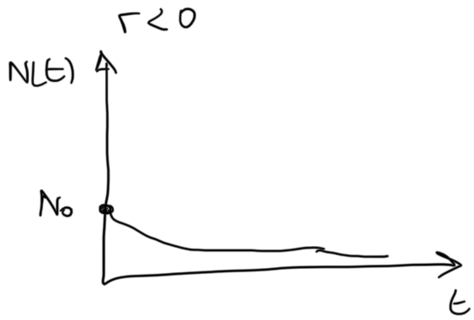
$$N(0) = C \cdot \underbrace{e^{r \cdot 0}}_{=1} = C = N_0 .$$

Therefore, the solution to $(*)$ is

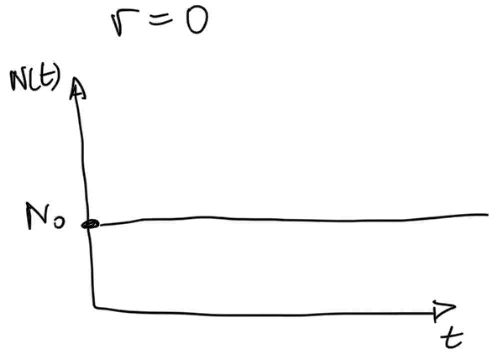
$$\boxed{N(t) = N_0 e^{rt}}$$



increasing population



decreasing population



population remains at level N_0

When $r > 0$, the solution $N(t)$ satisfies

$$\lim N(t) = \lim N_0 \cdot e^{rt} = +\infty$$

$t \to \infty \qquad t \to \infty$

When is this model observed? Whenever a population (animals, bacteria, etc.) can grow and multiply without any restrictions. (food, competition etc.).

When $r < 0 \Rightarrow$ "exponential decay" (eg. radioactive material).

II Restricted Growth

The von Bertalanffy Equation:

$$\left.\begin{aligned} \frac{dL}{dt} &= k(A-L) \\ L(0) &= L_0 \end{aligned}\right\} (**)$$

(**) describes the simplest form for restricted growth and can be used to describe the growth of fish.

$L(t)$ ... length of the fish at time t

$A, k > 0$ ... constants

Assumption: $L_0 < A$.

Solve (**):

$$\frac{dL}{dt} = k(A-L) \Rightarrow \int \frac{1}{A-L}\, dL = \int k\, dt$$

$$\Rightarrow -\ln|A-L| = kt + C$$

$$\Rightarrow \quad \ln|A - L| = -kt + c$$

$$\Rightarrow \quad A - L = e^{-kt} \cdot C$$

From $L(0) = L_0$, we get

$$A - L_0 = \underbrace{e^{-k \cdot 0}}_{=1} \cdot C \quad \Rightarrow C = A - L_0 .$$

The solution of (**) is:

$$A - L(t) = e^{-kt}(A - L_0)$$

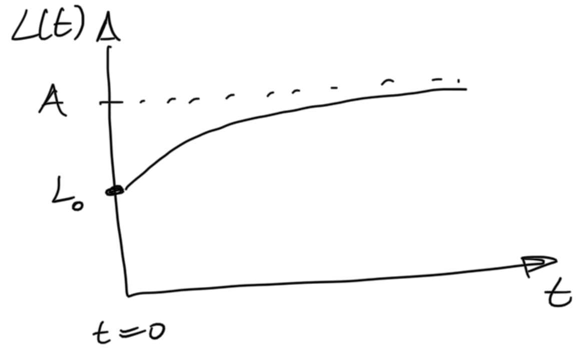
$$\Rightarrow \underline{\underline{L(t)}} = A - (A - L_0) \cdot e^{-kt}$$

$$= \underline{\underline{A\left[1 - \left(1 - \frac{L_0}{A}\right) e^{-kt}\right]}} .$$

$$\lim_{t \to \infty} L(t) = \lim_{t \to \infty} A\left[1 - \underline{\left(1 - \frac{L_0}{A}\right)} \cdot \underbrace{e^{-kt}}_{\to 0}\right]$$

$$= A .$$

$\Rightarrow A =$ "asymptotic length of the fish"



The DE $\quad \frac{dL}{dt} = k(A - L)$

[illegible] the fact that the growth rate express the ...

$\frac{dL}{dt}$ is proportional to the diffvence $A-L$ of the length of the fish from its asymptotic length.

- When L is smalle, $A-L$ is bigge, so $\frac{dL}{dt}$ is bigge (i.e., the fish grows faste)
- When L is bigger, $A-L$ is smalle, so $\frac{dL}{dt}$ is smalle (i.e., the fish grows more slowly).

The constant $k>0$ is just a proportionality constant.