Differential Equations

Introduction

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Differential Equations

A differential equation (DE) is an equation that involves a function $y = f(x)$ and some of its derivatives $y' = f'(x)$, $y'' = f''(x)$, ... and has to be solved w.r.t. $f(x)$.

When we solve an algebraic equation, we seek a number or a collection of numbers. When we solve a DE, we look for a *function*.

The order of a differential equation is defined to be that of the highest order derivative it contains. We will only deal with first order DEs, i.e., DEs which include only $y' = f'(x)$ and no higher derivatives.

Example: $f'(x) = 2f(x)$.

Differential Equations

A DE can be written in many different ways, for example:

$$f'(x) = 2f(x)$$
$$y' = 2y$$
$$\frac{dy}{dx} = 2y.$$

Differential Equations

In applications, the function f is some physical quantity and $\frac{df}{dx}$ is the **growth rate** of f (or rate of change of f) w.r.t. the variable x.

Example:

- t... time
- $N(t)$... the size of some population (e.g. mice)
- $\frac{dN}{dt}$... the growth rate of the population.

Solving Differential Equations

We focus here on **separable** DEs, that is, DEs of the form

$$\boxed{\frac{dy}{dx} = f(x) \cdot g(y).}$$

Definition

A DE is called **separable** if, by elementary algebraic manipulation, it is possible to arrange the equation s.t. all dependent variables (usually y) are on one side and all independent variables (usually x) are on the other side.

The corresponding solution technique is called **separation of variables**.

Solving Differential Equations: Separation of Variables

How it works:

$$y' = \frac{dy}{dx} = f(x) \cdot g(y)$$
$$\Longrightarrow \frac{dy}{g(y)} = f(x)dx \quad \text{(for } g(y) \neq 0\text{)}$$
$$\Longrightarrow \int \frac{1}{g(y)}dy = \int f(x)dx$$

and we calculate the integrals in order to solve the DE.

Solving Differential Equations: Separation of Variables

Example

Solve the DE $y' = (y^2+1)\cdot x$.

Solution:

$$\begin{aligned} y' = \frac{dy}{dx} &= (y^2+1)\cdot x \\ \implies \frac{dy}{(y^2+1)} &= x dx \\ \implies \int \frac{1}{(y^2+1)} dy &= \int x dx \\ \implies \arctan(y) &= \frac{1}{2}x^2 + c \\ \implies \tan\left(\arctan(y)\right) &= \tan\left(\frac{1}{2}x^2 + c\right) \\ \implies y(x) &= \tan\left(\frac{1}{2}x^2 + c\right), \quad c \in \mathbb{R}. \end{aligned}$$

Solving Differential Equations: Separation of Variables

Example

Solve $y' = (y+1)\cos(x)$.

Solution:

$$y' = \frac{dy}{dx} = (y+1)\cos(x) \quad \Longrightarrow \quad \frac{1}{y+1}dy = \cos(x)dx.$$

Case 1: If $y+1=0$, then $y=-1$, which solves the DE.
Case 2: If $y+1 \neq 0$, then we can proceed with $\frac{1}{y+1}dy = \cos(x)dx$:

$$\int \frac{1}{y+1}dy = \int \cos(x)dx \quad \Longrightarrow \quad \ln|y+1| = \sin(x) + c_1, \quad c_1 \in \mathbb{R}$$

$$\Longrightarrow |y+1| = \exp(\sin(x)+c_1) = \underbrace{\exp(\sin(x))}_{>0} \cdot \underbrace{\exp(c_1)}_{=:c}, \quad c \geq 0.$$

Solving Differential Equations: Separation of Variables

Resolving the absolute value yields

$y + 1 = c \cdot \exp(\sin(x)) \quad (c \geq 0)$ and $-(y+1) = c \cdot \exp(\sin(x)) \quad (c \geq 0)$.

We get

$$y = c \cdot \exp(\sin(x)) - 1 \text{ and } y = -c \cdot \exp(\sin(x)) - 1 \text{ for } c \geq 0.$$

This yields

$$y = c \cdot \exp(\sin(x)) - 1 \text{ for } c \in \mathbb{R}.$$

Note that if $c = 0$, we recover the first obtained solution $y = -1$.

Initial Value Problems

Sometimes, a DE is given together with an initial condition. Then we say that we have an **initial value problem** (IVP).

Example

Solve the IVP $y' = 2xy$, $y(0) = 1$.

Solution:
Step 1: Solve the DE:

$$y' = \frac{dy}{dx} = 2xy$$

$$\implies \frac{dy}{y} = 2xdx \quad \text{(for } y \neq 0\text{)}$$

$$\implies \int \frac{1}{y}dy = \int 2xdx$$

$$\implies \ln|y| = x^2 + c_1$$

$$\implies |y| = \exp(\ln|y|) = \exp(x^2 + c_1) = \exp(x^2) \cdot \underbrace{\exp(c_1)}_{=:c_2}$$

$$\implies y(x) = \exp(x^2) \cdot c_2, \quad c_2 \geq 0 \text{ and } -y(x) = -\exp(x^2) \cdot c_2, \quad c_2 \geq 0.$$

Initial Value Problems

Therefore, we get $y(x) = \exp(x^2) \cdot c$ with $c \in \mathbb{R}$.

Step 2: Solve the initial value problem:

$$y(0) = 1 = \exp(0) \cdot c = c \Longrightarrow c = 1.$$

Therefore, the solution of the IVP is

$$y(x) = \exp(x^2).$$

Whenever we solve an IVP, we will be able to find the constant c in the solution of the DE.

Initial Value Problems

Example

Solve the IVP $y' = 2x\sin(x^2)y^2$, $y(0) = \frac{1}{2}$.

Solution:
Step 1: Solve the DE:

$$\begin{aligned} y' = \frac{dy}{dx} &= 2x\sin(x^2)y^2 \\ \implies \frac{dy}{y^2} &= 2x\sin(x^2)dx \quad \text{(for } y \neq 0) \\ \implies \int \frac{1}{y^2}dy &= \int 2x\sin(x^2)dx \\ \implies -\frac{1}{y} &= -\cos(x^2) + c, \quad c \in \mathbb{R} \\ \implies y &= \frac{1}{\cos(x^2) - c}, \quad c \in \mathbb{R}. \end{aligned}$$

Because of $y(0) = \frac{1}{2}$, $y = 0$ cannot be a solution of the IVP.

Initial Value Problems

Step 2: Solve the initial value problem:

$$y(0) = \frac{1}{\cos(0^2) - c} = \frac{1}{1 - c} = \frac{1}{2}.$$

We have $1 - c = 2$ and therefore, $c = -1$. Therefore, the solution of the IVP is

$$y(x) = \frac{1}{\cos(x^2) + 1}.$$