Lecture 4

Partial Fraction Decomposition

$$f(x) = \frac{P(x)}{Q(x)}$$

1. Linear Factors

Case 1a) $Q(x)$ is a product of m distinct linear factors:

$$Q(x) = a \cdot (x - x_1) \cdot (x - x_2) \cdot \ldots \cdot (x - x_m)$$

$\left(\text{or: } Q(x) = (a_1 x + b_1) \cdot \ldots \cdot (a_m x + b_m) \right)$

where $x_1, \ldots, x_m$ are the m distinct roots of $Q(x)$. Then

$$\frac{P(x)}{Q(x)} = \frac{1}{a} \cdot \left[\frac{A_1}{x - x_1} + \ldots + \frac{A_m}{x - x_m} \right]$$

$$\left(\text{or: } \frac{P(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \ldots + \frac{A_m}{a_m x + b_m} \right)$$

Case 1b) $Q(x)$ is a product of repeated linear factors:

$$Q(x) = a(x - x_1)^r$$

(or: $Q(x) = (\tilde{a}x + b)^r$)

where x_1 is the root of $Q(x)$.

Then $\qquad x_1 = 0$

$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left[\frac{A_1}{x-x_1} + \frac{A_2}{(x-x_1)^2} + \dots + \frac{A_r}{(x-x_1)^r}\right]$$

(or: $\dfrac{P(x)}{Q(x)} = \dfrac{A_1}{\tilde{a}x+b} + \dfrac{A_2}{(\tilde{a}x+b)^2} + \dots + \dfrac{A_r}{(\tilde{a}x+b)^r}$)

Example: Evaluate $\displaystyle\int \frac{dx}{x^2(x+1)}$

The function $Q(x) = x^2(x+1)$ is a product of three linear functions: $\underbrace{x, x}_{\text{repeated}}, \underbrace{x+1}_{\text{distinct}}$

Hence, we can write

$$\frac{\boxed{1}}{x^2(x+1)} = \boxed{\frac{A_1}{x} + \frac{A_2}{x^2}} + \frac{A_3}{x+1}$$

$x, x, x+1$

$$= \frac{\boxed{A_1 x(x+1)}}{x^2(x+1)} + \frac{\boxed{A_2(x+1)}}{x^2(x+1)} + \frac{\boxed{A_3 \cdot x^2}}{x^2(x+1)}$$

$$\Rightarrow 1 = A_1 x(x+1) + A_2(x+1) + A_3 x^2$$
$$= (A_1 + A_3)x^2 + (A_1 + A_2)x + A_2$$

$\Rightarrow A_2 = 1 \qquad \Updownarrow \quad A_1 + A_3 = 0 \Rightarrow -1 + A_3 = 0$

$A_1 + A_2 = 0 \Rightarrow A_1 + 1 = 0$

$\Rightarrow A_1 = -1$

$A_3 = 1$

$$\Rightarrow \frac{1}{x^2(x+1)} = -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$$

$$\Rightarrow \int \frac{1}{x^2(x+1)}\,dx = \int \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right)dx$$

$$= -\ln|x| - \frac{1}{x} + \ln|x+1| + C$$

Irreducible Polynomials of Degree 2

A polynomial of degree 2 that cannot be factored into a product of two linea functions with real coefficients is called <u>irreducible</u>.

Example:
- x^2+1 is irreducible, because it cannot be represented as $(ax+b)(cx+d)$ with $a,b,c,d \in \mathbb{R}$.
- x^2-1 is reducible, as $x^2-1 = (x-1)(x+1)$.

Irreducible Quadratic Factors $f(x) = \frac{P(x)}{\ldots}$

Irreducible quadratic ... $Q(x)$

If the irreducible quadratic factor ax^2+bx+c is contained n times in the factorization of the denominator of a proper rational function, then

$$\boxed{\frac{P(x)}{Q(x)} = \boxed{\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2}} + \dots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}}$$

$n=2$

Examples

(i) Solve $\displaystyle\int \frac{2x^3-x^2+2x-2}{(x^2+2)(x^2+1)}\,dx$

The function $Q(x)=(x^2+2)(x^2+1)$ is a product of two quadratic functions that are irreducible, and the two factors are distinct.

Hence:

$$\frac{\boxed{2x^3-x^2+2x-2}}{(x^2+2)(x^2+1)} = \frac{B_1x+C_1}{x^2+2} + \frac{D_1x+E_1}{x^2+1}$$

$$= \frac{\boxed{(B_1x+C_1)(x^2+1)}}{(x^2+2)(x^2+1)} + \frac{\boxed{(D_1x+E_1)(x^2+2)}}{(x^2+2)(x^2+1)}$$

$$\Rightarrow \boxed{2x^3 - x^2 + 2x - 2} = (B_1 x + C_1)(x^2+1) + (D_1 x + E_1)(x^2+2)$$

$$= B_1 x^3 + B_1 x + C_1 x^2 + C_1 + D_1 x^3 + 2D_1 x + E_1 x^2 + 2E_1$$

$$= \boxed{(B_1 + D_1)x^3 + (C_1 + E_1)x^2 + (B_1 + 2D_1)x + C_1 + 2E_1}$$

$$\Rightarrow \boxed{2 = B_1 + D_1} \Rightarrow B_1 + D_1 = B_1 + D_1 + D_1 = 2,$$

$$-1 = C_1 + E_1$$

$$\boxed{2 = B_1 + 2D_1}$$

$$-2 = C_1 + 2E_1$$

hence $\underline{\underline{D_1 = 0}}$, $\underline{\underline{B_1 = 2}}$

$$\to \underbrace{C_1 + E_1}_{=-1} + E_1 = -2$$

$$\Rightarrow -1 + E_1 = -2 \Rightarrow \underline{\underline{E_1 = -1}}$$

$$\to C_1 = -1 - E_1 = -1 - (-1) = 0$$

$$\Rightarrow \underline{\underline{C_1 = 0}}$$

Hence :

$$\frac{2x^3 - x^2 + 2x - 2}{(x^2+2)(x^2+1)} = \frac{2x}{x^2+2} - \frac{1}{x^2+1}$$

$$\int \frac{2x^3 - x^2 + 2x - 2}{} dx = \int \frac{2x}{x^2+2} \boxed{dx} - \int \frac{1}{x^2+1} d$$

$$(x^2+2)(x^2+1)$$

substitution: $u = x^2+2 \Rightarrow \frac{du}{dx} = 2x$

$$\Rightarrow dx = \boxed{\frac{du}{2x}}$$

$$= \int \frac{\cancel{2x}}{u} \frac{du}{\cancel{2x}} - \int \frac{1}{x^2+1} dx$$

$$= \int \frac{1}{u} du - \int \frac{1}{x^2+1} dx$$

$$= \ln|u| - \arctan x + c$$

$$= \ln|x^2+2| - \arctan x + C$$

(ii) Evaluate $\int \frac{x^2+x+1}{(x^2+1)^2} dx$

The function $\frac{x^2+x+1}{(x^2+1)^2}$ is proper, since the degree of the polynomial in the numerator is 2, and the degree of the polynomial in the denominator is 4 ($2<4 \Rightarrow$ proper).

Therefore:

$$\frac{\boxed{x^2+x+1}}{(x^2+1)^2} = \frac{B_1x+C_1}{x^2+1} + \frac{B_2x+C_2}{(x^2+1)^2}$$

$$= \frac{(B_1x+C_1)(x^2+1)}{(x^2+1)^2} + \frac{B_2x+C_2}{(x^2+1)^2}$$

$$B_1x^3 + B_1x + C_1x^2 + C_1 + B_2x + C_2$$

$$= \frac{\dots}{(x^2+1)^2}$$

$$= \frac{\boxed{B_1 x^3 + C_1 x^2 + (B_1+B_2)x + C_1+C_2}}{(x^2+1)^2}$$

$$\Rightarrow x^2 + x \boxed{+1} = B_1 x^3 + C_1 x^2 + (B_1+B_2)x + \boxed{C_1+C_2}$$

$\Rightarrow B_1 = 0$, since x^3 does not appear on the left-hand side.

$\Rightarrow C_1 = 1$

$\Rightarrow 1 = B_1 + B_2 \Rightarrow B_2 = 1$

$\Rightarrow 1 = C_1 + C_2 = 1 + C_2 \Rightarrow C_2 = 0$

Hence, we get

$$\frac{x^2+x+1}{(x^2+1)^2} = \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}$$

$$\Rightarrow \int \frac{x^2+x+1}{(x^2+1)^2}\,dx = \underbrace{\int \frac{1}{x^2+1}\,dx}_{=\arctan(x)+\tilde{c}} + \int \frac{x}{(x^2+1)^2}\,dx$$

$$\int \frac{x}{(x^2+1)^2}\,dx \underset{\uparrow}{=} \int \frac{\cancel{x}}{u^2}\frac{du}{2\cancel{x}} = \frac{1}{2}\int \frac{1}{u^2}\,du$$

Substitution: $u = x^2+1$

$\Rightarrow \frac{du}{dx} = 2x$

$\Rightarrow dx = \frac{du}{2x}$

$$= \frac{1}{2}\left(-\frac{1}{u}\right) + \overline{c}$$

$$= -\frac{1}{2}\cdot\frac{1}{x^2+1} + \bar{c}$$

In total:

$$\int \frac{x^2+x+1}{(x^2+1)^2}\,dx = \underline{\underline{\arctan(x) - \frac{1}{2}\cdot\frac{1}{x^2+1} + c}}$$

(with $\tilde{c}+\bar{c} = c$)

<u>Example</u>: Evaluate $\int \frac{x^4-5x^3+6x^2-18}{x^3-3x^2}\,dx$

$\Rightarrow$ degree of the numerator $= 4 > 3$ $=$ degree of denum

$\Rightarrow$ Thus, the rational function is <u>not</u> proper.

$\Rightarrow$ We perform a long division:

$$(x^4-5x^3+6x^2-18) : (x^3-3x^2) = \underline{\underline{x-2 - \frac{18}{x^3-3x^2}}}$$

$$\begin{array}{l} -\underline{(x^4-3x^3)} \\ \quad 0 - 2x^3 \\ \quad -\underline{(-2x^3+6x^2)} \\ \qquad 0 + 0 - 18 \end{array}$$

- We perform a partial fraction decomposition for $\frac{18}{x^3-3x^2}$:

$$\frac{18}{x^3-3x^2} = \frac{18}{x^2(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}$$

$$= \frac{Ax(x-3) + B(x-3) + Cx^2}{x^2(x-3)}$$

$$= \frac{Ax^2 - 3Ax + Bx - 3B + Cx^2}{x^2(x-3)}$$

$$\Rightarrow 18 = Ax^2 - 3Ax + Bx - 3B + Cx^2$$

Since this equation must hold for all x, we get $18 = -3B \Rightarrow \underline{\underline{B = -6}}$

Moreover, since x or x^2 do not appear on the left-hand side of (*), we obtain

$$0 = Ax^2 - 3Ax + Bx + Cx^2$$
$$= \underbrace{(A+C)}_{=0}x^2 + \underbrace{(B-3A)}_{=0}x$$

$$\Rightarrow \begin{cases} A + C = 0 \\ B - 3A = 0 \overset{B=-6}{\Longrightarrow} -6 - 3A = 0 \\ \qquad \Rightarrow -3A = 6 \\ \qquad \Rightarrow \underline{\underline{A = -2}} \end{cases}$$

$$\Rightarrow A + C = -2 + C = 0 \Rightarrow \underline{\underline{C = 2}}$$

Therefore, we have for our integral:

$$\int \frac{x^4 - 5x^3 + 6x^2 - 18}{x^3 - 3x^2}\,dx$$

$$= \int (x-2)\,dx - \int \left(-\frac{2}{x} - \frac{6}{x^2} + \frac{2}{x-3}\right) d$$

$$= \frac{1}{2}x^2 - 2x + \ln|x| - \frac{6}{x} - 2\ln|x-3| + c$$

Short Summary

$$f(x) = \frac{P(x)}{Q(x)}$$

P, Q ... polynomials

degree of $P(x)$ < degree of $Q(x)$

(otherwise: perform long division)

(recall that degree = largest exponent in the polynomial)

Factor in denominator	Term in Partial Fraction Decomp.
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^r$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_r}{(ax+b}$
ax^2+bx+c	$\frac{Bx+C}{ax^2+bx+c}$

$$(ax^2+bx+c)^r$$

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c}$$

$$+ \ldots + \frac{B_rx+C_r}{(ax^2+bx+c)^r}$$