Partial Fraction Decomposition

Linear Factors

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Rational Function

A **rational function** f is the quotient of two polynomials. That is,

$$f(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials, i.e., functions of the form $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$.

Proper Rational Function

A rational function is called **proper** if the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator.

Otherwise the rational function is called **improper**.

Long Division

If f is an **improper** rational function, then the first step in the partial-fraction decomposition is to use **long division** to write f as a sum of a polynomial and a **proper** rational function.

Long Division – Example 1

Example 1

Find $\int \frac{x}{x+2} dx$.

The function $f(x) = \frac{x}{x+2}$ is improper, as the degree of the numerator is equal to the degree of the denominator $\Longrightarrow$ use long division:

Long Division – Example 1

This is a polynomial and a **proper** rational function. We can integrate the integrand in this new form:

$$\int \frac{x}{x+2} dx = \int \left(1 - \frac{2}{x+2}\right) dx = x - 2\ln|x+2| + C.$$

Long Division – Example 2

Example 2

Find $\int \frac{3x^3-7x^2+17x-3}{x^2-2x+5}dx$.

The function $f(x) = \frac{3x^3-7x^2+17x-3}{x^2-2x+5}$ is improper, as the degree of the numerator is higher than the degree of the denominator $\Longrightarrow$ use long division:

Long Division – Example 2

Now, by the quadratic formula, $x^2 - 2x + 5 = 0$ does not have real solutions. This means that it is irreducible, so we will complete the square instead and obtain

$$x^2 - 2x + 5 = (x^2 - 2x + 1) + 4 = (x - 1)^2 + 4.$$

The integral we wish to evaluate is therefore

$$\int \left(3x - 1 + \frac{2}{(x-1)^2 + 4}\right) dx = \int (3x - 1) dx + 2 \int \frac{1}{(x-1)^2 + 4} dx.$$

The first integral on the right-hand side is straightforward since the integrand is a polynomial; we find that

$$\int (3x - 1) dx = \frac{3}{2}x^2 - x + C_1.$$

Long Division – Example 2

To evaluate the second integral, we use the following trick:

$$\int \frac{1}{(x-1)^2+4}dx = \frac{1}{4}\int \frac{1}{1+(\frac{x-1}{2})^2}dx.$$

Now we substitute: $u(x) = \frac{x-1}{2}$ with $\frac{du}{dx} = \frac{1}{2}$ and thus $dx = 2du$. We obtain:

$$\begin{aligned}\frac{1}{4}\int \frac{1}{1+(\frac{x-1}{2})^2}dx = \frac{1}{4}\int \frac{1}{1+u^2}2du = \frac{1}{2}\int \frac{1}{1+u^2}du \\ = \frac{1}{2}\arctan u + C_2 \\ = \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + C_2.\end{aligned}$$

In conclusion, we get

$$\int \frac{3x^3-7x^2+17x-3}{x^2-2x+5}dx = \frac{3}{2}x^2 - x + \arctan\left(\frac{x-1}{2}\right) + C.$$

Irreducible Polynomials

A polynomial that cannot be factored into a product of two linear functions with real coefficients is called **irreducible**. Otherwise it is called **reducible**.

Example:

1. $x^2 + 1$ is irreducible, as it cannot be represented as $(ax + b)(cx + d)$ with a, b, c, d as real numbers.
2. $x^2 - 1$ is reducible, as $x^2 - 1 = (x - 1)(x + 1)$.

Partial Fraction Decomposition

Partial fraction decomposition means rewriting a rational function $f(x) = \frac{P(x)}{Q(x)}$ (with P, Q being polynomials) as the sum of a polynomial and a simpler rational function of the type

$$\frac{A}{(ax+b)^n} \quad \text{or} \quad \frac{Bx+C}{(ax^2+bx+c)^n}.$$

Partial Fraction Decomposition – Linear Factors

Case 1: $Q(x)$ is the product of m distinct linear factors

If $Q(x)$ is of the form

$$Q(x) = a(x - x_1)(x - x_2) \cdots (x - x_m),$$

where $x_1, \ldots, x_m$ are the m distinct roots of $Q(x)$, then there exist constants $A_1, \ldots, A_m$ with

$$\frac{P(x)}{Q(x)} = \frac{1}{a} \left(\frac{A_1}{x - x_1} + \frac{A_2}{x - x_2} + \ldots + \frac{A_m}{x - x_m} \right).$$

Partial Fraction Decomposition – Distinct Linear Factors

Example

Find $\int \frac{1}{x(x-1)}dx$.

We set

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)+Bx}{x(x-1)} = \frac{(A+B)x-A}{x(x-1)}.$$

Hence, we get $1 = (A+B)x - A$. Since this equation must hold for any x, it can only be true if $A+B=0$ and $1=-A$. This yields $A=-1$ and $B=1$. Hence,

$$\frac{1}{x(x-1)} = \frac{-1}{x} + \frac{1}{x-1}$$

and

$$\int \frac{1}{x(x-1)}dx = \int \left(\frac{-1}{x} + \frac{1}{x-1}\right)dx = -\ln|x| + \ln|x-1| + C$$

$$= \ln|x-1| - \ln|x| + C = \ln\left|\frac{x-1}{x}\right| + C.$$

Partial Fraction Decomposition – Linear Factors

Case 2: $Q(x)$ is the product of r identical linear factors

If $Q(x)$ is of the form

$$Q(x) = a(x - x_1)^r,$$

where x_1 is the root of $Q(x)$, then there exist constants $A_1, \ldots, A_r$ with

$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left(\frac{A_1}{x - x_1} + \frac{A_2}{(x - x_1)^2} + \ldots + \frac{A_r}{(x - x_1)^r}\right).$$

Partial Fraction Decomposition – Identical Linear Factors

Example

Find $\int \frac{x}{(x+1)^2}\,dx$.

Here we can write

$$\frac{x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1)+B}{(x+1)^2} = \frac{Ax+(A+B)}{(x+1)^2}.$$

We obtain $A = 1$ and $A + B = 0$, hence $B = -1$. So, we get

$$\int \frac{x}{(x+1)^2}dx = \int \left(\frac{1}{(x+1)} - \frac{1}{(x+1)^2} \right) dx$$
$$= \ln|x+1| + \frac{1}{x+1} + C.$$