Revision Lecture - Previous Exam (2021)

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Question 1 A

Evaluate the integral

$$\int_0^1 \frac{x}{x^2+2}\,\boxed{dx.}$$

Solution: Substitution: $\boxed{u = x^2+2}$, $u' = \frac{du}{dx} = 2x$

$\Rightarrow \boxed{dx = \frac{du}{2x}}$

$$\Rightarrow \int_0^1 \frac{x}{x^2+2}\,dx = \int_2^3 \frac{\cancel{x}}{u}\,\frac{du}{2\cancel{x}}$$

| limits: $x = 0 \Rightarrow u = 2$
$x = 1 \Rightarrow u = 3$

$$= \frac{1}{2}\int_2^3 \frac{1}{u}\,du$$

$$= \frac{1}{2}\Big[\ln u\Big]_2^3$$

$$= \frac{1}{2}(\ln 3 - \ln 2)$$

$$= \frac{1}{2}\ln\frac{3}{2}$$

Question 1 B

We are given the following matrices:

$$A = \begin{pmatrix} 2 & -1 \\ -7 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & \boxed{4} \\ \boxed{2} & 10 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}.$$

1. Which of the matrices A, B and C is invertible?
2. Calculate $B + A^{-1}$.
3. Given the vectors $b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $c = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Are b and c orthogonal?

1. $\det A = 2 \cdot 3 - (-1) \cdot (-7) = 6 - 7 = -1 \neq 0 \Rightarrow A$ is invertible
$\det B = (-1) \cdot 10 - 4 \cdot 2 = -10 - 8 = -18 \neq 0 \Rightarrow B$ is invertible
$\det C = 1 \cdot 4 - (-2) \cdot (-2) = 4 - 4 = 0 \Rightarrow C$ is not invertible.

2. $A^{-1} = \frac{1}{\det A} \cdot \begin{pmatrix} 3 & 1 \\ 7 & 2 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ -7 & -2 \end{pmatrix}$ (directly by the formula for the inverse)

Or: $A \cdot A^{-1} = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 2 & -1 \\ -7 & 3 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \begin{aligned} 2\cdot a + (-1)\cdot c &= 1 \\ 2\cdot b + (-1)\cdot d &= 0 \\ -7a + 3c &= 0 \\ -7b + 3d &= 1 \end{aligned}$$

$\Longrightarrow c = 2a - 1$

$\Downarrow$

$\Longrightarrow -7a + 3(2a-1) = 0$

$-7a + 6a - 3 = 0$

$\Longrightarrow \underline{\underline{a = -3}}$

$c = 2\cdot(-3) - 1 = \underline{\underline{-7}}$

$\to d = 2b$

$\to -7b + 3\cdot 2b = 1$

$\Longrightarrow \underline{\underline{b = -1}}$

$d = 2\cdot(-1) = \underline{\underline{-2}}$

$$\Longrightarrow A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ -7 & -2 \end{pmatrix}$$

$$B + A^{-1} = \begin{pmatrix} -1 & 4 \\ 2 & 10 \end{pmatrix} + \begin{pmatrix} -3 & -1 \\ -7 & -2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -4 & 3 \\ -5 & 8 \end{pmatrix}}}$$

3. $b^T c = b \cdot c = -1\cdot 1 + 2\cdot(-2) = -5 \neq 0$

$\to$ b and c are <u>not</u> orthogonal.

Question 2

A group of biologists have observed lions in the jungle. They concluded that the number of individuals $N(t)$ at time $t > 0$ is given by the differential equation

$$\boxed{\frac{dN(t)}{dt} = \frac{N(t)}{1000}(500 - N(t)).}$$

$t=0$

1. Find $N(t)$ given that the population is $N(0) = 50$ at time $t = 0$.
2. What does the differential equation say about the population's per capita growth rate? What is the carrying capacity of the population?
3. Find the equilibrium points of the population growth model described above and classify them according to their stability.
4. Some biologists from the group that studied the population of lions have suggested that the growth model described above is not realistic for the following reason: it does not take into account the fact that when the number $N(t)$ of lions is very small, the population will decline because of the lack of mates for reproduction. Can you suggest a variant of the given differential equation that takes into account the fact that when the population of lions is less than 40 then the population will decrease to 0?

1. We solve the DE $\frac{dN}{dt} = \frac{N}{1000}(500-N)$

separation of variable $\Longrightarrow$ $\int \frac{dN}{N(500-N)} = \int \frac{1}{1000}\,dt$

Partial Fraction decomposition $\Longrightarrow$ $\int \frac{1}{500}\cdot\left(\frac{1}{N} + \frac{1}{500-N}\right) dN = \frac{1}{1000}\int dt$

$\Longrightarrow \frac{1}{500}\left(\int \frac{1}{N}\,dN + \int \frac{1}{500-N}\,dN\right) = \frac{1}{1000}\int dt \quad \Big|\cdot 500$

$\Longrightarrow \int \frac{1}{N}\,dN + \int \frac{1}{500-N}\,dN = \frac{1}{2}\int dt$

$\Longrightarrow \ln N + \left(-\ln(500-N)\right) = \frac{1}{2}t + c \qquad (c: \text{constant})$

$\Longrightarrow \ln \frac{N}{500-N} = \frac{1}{2}t + c$

$\Longrightarrow \frac{N}{500-N} = e^{\frac{1}{2}t}\cdot\tilde{c} \qquad (\ )^{-1}$

$\frac{a}{b} = \frac{c}{d}$

$\Longrightarrow \frac{b}{a} = \frac{d}{c}$

$\Longrightarrow \frac{500-N}{N} = \frac{500}{N} - 1 = \frac{1}{e^{\frac{1}{2}t}\cdot\tilde{c}}$

$\Longrightarrow \frac{500}{N} = \frac{1}{e^{\frac{1}{2}t}\tilde{c}} + 1 = \frac{1 + e^{\frac{1}{2}t}\tilde{c}}{e^{\frac{1}{2}t}\cdot\tilde{c}}$

$$\Rightarrow N = \frac{500 \cdot e^{\frac{1}{2}t} \cdot \tilde{c}}{1 + e^{\frac{1}{2}t}\tilde{c}}$$

Initial condition: $N(0) = 50$

$$N(0) = 50 = \frac{500 \cdot e^{\frac{1}{2}\cdot 0} \cdot \tilde{c}}{1 + e^{\frac{1}{2}\cdot 0} \cdot \tilde{c}} = \frac{500 \cdot \tilde{c}}{1 + \tilde{c}}$$

$$50 \cdot (1 + \tilde{c}) = 500\tilde{c}$$
$$50 + 50\tilde{c} = 500\tilde{c} \quad \Rightarrow \quad 50 = 450\tilde{c} \quad \Rightarrow \quad \tilde{c} = \frac{50}{450} = \underline{\underline{\frac{1}{9}}}$$

$$\Rightarrow N(t) = \frac{500 \cdot e^{\frac{1}{2}t} \cdot \frac{1}{9}}{1 + e^{\frac{1}{2}t} \cdot \frac{1}{9}}$$

$$= \frac{500\, e^{\frac{1}{2}t}}{9 + e^{\frac{1}{2}t}} \quad \Big| \cdot e^{-\frac{1}{2}t}$$

$$= \underline{\underline{\frac{500}{9e^{-\frac{1}{2}t} + 1}}}$$

2. The per capita growth rate is

$$\frac{dN/dt}{N} = \frac{1}{1000}(500-N),$$ so it depends linearly on N.

The bigger N is, the smaller the per capita growth rate.

The given DE is a logistic DE, i.e., of the form

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$$

with $r = \frac{1}{2}$ and $K = 500$.

The carrying capacity of the population is 500.

3. Set $g(N) = \frac{N}{1000} \cdot (500-N) = 0$

$\Rightarrow N = 0$ and $N = 500$ are equilibria.

$g'(N) = \frac{1}{2} - \frac{N}{500}$

$g'(0) = \frac{1}{2} > 0 \Rightarrow N = 0$ is an unstable equ.

$g'(500) = \frac{1}{2} - 1 = -\frac{1}{2} < 0 \Rightarrow N = 500$ is a loc. stable equ.

4. $\frac{dN}{dt} = \frac{N}{1000}\underline{(N-40)}(500-N)$

Question 3

The growth of a population of mice (krattspissmus) is described by the following Leslie matrix

$$\boxed{L = \begin{pmatrix} 0 & 5 \\ 0.8 & 0 \end{pmatrix}.}$$

1. What information does the matrix L provide about the population? (age groups, offspring rate, survival rates)
2. Find the eigenvectors and eigenvalues of L.
3. What is the biological interpretation of the larger eigenvalue of L?
4. What is the percentage of each age group in the population after a large number of reproduction sessions? Justify your answer.

1. The population of females consists of two age groups, the 0-year olds and the 1-year olds.

$$L = \begin{pmatrix} F_0 & F_1 \\ P_0 & 0 \end{pmatrix}:$$

F_0: offspring rate of 0-year olds $= 0$: 0-year olds do not produce offspring

F_1: — " — 1- —" - $= 5$: Each female 1-year old produces on average 5 offspring.

P_0: Survival rate of 0-year olds $= 0.8$: On average, 80 % of the zero-year olds survive.

2. $\det(\lambda I_2 - L) = \det\begin{pmatrix} \lambda & -5 \\ -0.8 & \lambda \end{pmatrix} = \lambda^2 - 5 \cdot 0.8 = \lambda^2 - 4 = 0$

$\Rightarrow \lambda_1 = 2, \quad \lambda_2 = -2$ are the eigenvalues of L.

· eigenvector corr. to $\lambda_1 = 2$: $L \cdot u = 2 \cdot u$

$$\Rightarrow \begin{pmatrix} 0 & 5 \\ 0.8 & 0 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}_{=u} = \begin{pmatrix} 2u_1 \\ 2u_2 \end{pmatrix} \Rightarrow \begin{array}{l} 5 \cdot u_2 = 2u_1 \\ 0.8 u_1 = 2u_2 \end{array}$$

$\Rightarrow$ This is one equation $5u_2 = 2u_1$.

For example, $\boxed{u = \begin{pmatrix} 5 \\ 2 \end{pmatrix}}$ is an eigenvector corr. to λ_1.

· eigenvector corr. to $\lambda_2 = -2$: $Lv = -2 \cdot v$

$$\Rightarrow \begin{pmatrix} 0 & 5 \\ 0.8 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2v_1 \\ -2v_2 \end{pmatrix} \Rightarrow \begin{array}{l} 5v_2 = -2v_1 \\ 0.8v_1 = -2v_2 \end{array}$$

For example, $v = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ is an eigenvector corr. to λ_2.

3. The large eigenvalue of L expresses the asymptotic growth rate of the population. After sufficiently many years, the population at time $t+1$ will be approximately two times (λ_1-times) the population at time t.

4. We know that the eigenvector u which corresponds to the large eigenvalue is a stable age distribution. This means that after a large number of reproduction sessions, the population of mice will consist of approx.

$$\frac{5}{7} \approx 71\% \text{ of 0-year olds}$$

and $$\frac{2}{7} \approx 29\% \text{ of 1-year olds.}$$

$$\left(u = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Question 4

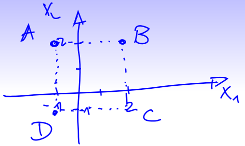
Consider the function

$$\boxed{f(x_1, x_2) = x_1^2 - 2x_1x_2 + x_2^3}$$

defined on the closed and bounded domain

$$\boxed{D = \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_1 \leq 2,\ -1 \leq x_2 \leq 2\}.}$$

1. Find the global maximum and minimum of f on the domain D.
2. Does f have a local maximum or minimum at the point $(0, 0)$? Does it have a local maximum or minimum at $(\frac{3}{2}, \frac{3}{2})$?
3. Find the equation of the tangent plane of the graph of f at the point $(1, 1)$.
4. Find the directional derivative of f at the point $(1, 1)$ towards the direction of the vector $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.



1. Look for local extrema of f in the interior of D:

$$\nabla f(x_1,x_2) = \begin{pmatrix} 2x_1 - 2x_2 \\ -2x_1 + 3x_2^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \boxed{x_1 = x_2}$$

$$-2x_1 + 3x_1^2 = x_1(-2+3x_1) = 0 \Longrightarrow x_1 = 0 \text{ or } x_1 = \frac{2}{3}$$

Hence, the critical points are $(0,0)$ with $\underline{f(0,0) = 0}$ and $\left(\frac{2}{3}, \frac{2}{3}\right)$ with $\underline{f\left(\frac{2}{3}, \frac{2}{3}\right) = -\frac{4}{27}}$.

Now, we find the values of f on the boundary of D:

- AB: $-1 \le x_1 \le 2$, $x_2 = 2$

$$f(x_1, 2) = x_1^2 - 4x_1 + 8 =: g(x_1)$$

$$\Longrightarrow g'(x_1) = 2x_1 - 4 = 0 \Longrightarrow x_1 = 2, \ \underline{f(2,2) = 4}.$$

$$\underline{f(-1,2) = 13}$$

- BC: $x_1 = 2$, $-1 \le x_2 \le 2$

$$f(2, x_2) = 4 - 4x_2 + x_2^3 =: g(x_2) \Longrightarrow g'(x_2) = -4 + 3x_2^2 = 0$$

$$\Rightarrow \quad x_2 = \sqrt{\frac{4}{3}} \quad \Rightarrow \quad f\left(2, \sqrt{\frac{4}{3}}\right) \approx \underline{0.9}$$

$$\Rightarrow \quad x_2 = -\sqrt{\frac{4}{3}} \quad \Rightarrow \quad f\left(2, -\sqrt{\frac{4}{3}}\right) \approx \underline{7.08}$$

$$\underline{f(2,-1) = 7}$$

• CD : $-1 \leq x_1 \leq 2, \quad x_2 = -1$

$$f(x_1, -1) = x_1^2 + 2x_1 - 1 =: g(x_1)$$

$$\Rightarrow g'(x_1) = 2x_1 + 2 \quad \Rightarrow \quad x_1 = -1 \quad \Rightarrow \quad f(1,-1) = \ldots\ldots$$

$$\underline{f(-1,-1) = -2}$$

• DA : $x_1 = -1, -1 \leq x_2 \leq 2$

$$f(-1, x_2) = 1 + 2x_2 + x_2^3 =: g(x_2)$$

$\Rightarrow g'(x_2) = 2 + 3x_2^2 = 0 \quad \Rightarrow$ no real solutions

$$\underline{f(-1,2) = 13}$$

$\Rightarrow$ Comparing all values on the boundary of D as well as the values of f at the critical point in the interior of D gives:

$(-1,-1)$ with $f(-1,-1) = -2$ is a global min.
$(-1,2)$ with $f(-1,2) = 13$ is a global max.