

Revision Lecture - Previous Exam (2021)

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Question 1 A

Evaluate the integral

$$\int_0^1 \frac{x}{x^2+2} dx.$$

Solution: Substitution:

$$u = x^2 + 2$$

$$u' = \frac{du}{dx} = 2x$$

$$\Rightarrow \int_0^1 \frac{x}{x^2+2} dx = \int_2^3 \frac{\cancel{x}}{u} \frac{du}{2x}$$

$$\Rightarrow \boxed{dx = \frac{du}{2x}}$$

$$= \frac{1}{2} \int_2^3 \frac{1}{u} du$$

limits: $x=0 \Rightarrow u=2$
 $x=1 \Rightarrow u=3$

$$= \frac{1}{2} [\ln u]_2^3$$

$$= \frac{1}{2} (\ln 3 - \ln 2)$$

$$= \underline{\underline{\frac{1}{2} \ln \frac{3}{2}}}}$$

Question 1 B

We are given the following matrices:

$$A = \begin{pmatrix} 2 & -1 \\ -7 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 4 \\ 2 & 10 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}.$$

1. Which of the matrices A , B and C is invertible?
2. Calculate $B + A^{-1}$.
3. Given the vectors $b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $c = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Are b and c orthogonal?

1. $\det A = 2 \cdot 3 - (-1) \cdot (-7) = 6 - 7 = -1 \neq 0 \Rightarrow A$ is invertible
 $\det B = (-1) \cdot 10 - 4 \cdot 2 = -10 - 8 = -18 \neq 0 \Rightarrow B$ is invertible
 $\det C = 1 \cdot 4 - (-2) \cdot (-2) = 4 - 4 = 0 \Rightarrow C$ is not invertible.

2. $A^{-1} = \frac{1}{\det A} \cdot \begin{pmatrix} 3 & 1 \\ 7 & 2 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ -7 & -2 \end{pmatrix}$ (directly by the formula for the inverse)

Or: $A \cdot A^{-1} = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 2 & -1 \\ -7 & 3 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow 2a + (-1) \cdot c = 1 \quad \Rightarrow c = 2a - 1$$

$$2b + (-1) \cdot d = 0$$

$$-7a + 3c = 0$$

$$-7b + 3d = 1$$

$$\Rightarrow -7a + 3(2a - 1) = 0$$

$$-7a + 6a - 3 = 0$$

$$\Rightarrow \underline{\underline{a = -3}}$$

$$c = 2 \cdot (-3) - 1 = \underline{\underline{-7}}$$

$$\Rightarrow d = 2b$$

$$\Rightarrow -7b + 3 \cdot 2b = 1$$

$$\Rightarrow \underline{\underline{b = -1}}$$

$$d = 2 \cdot (-1) = \underline{\underline{-2}}$$

$$\Rightarrow A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ -7 & -2 \end{pmatrix}$$

$$B + A^{-1} = \begin{pmatrix} -1 & 4 \\ 2 & 10 \end{pmatrix} + \begin{pmatrix} -3 & -1 \\ -7 & -2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -4 & 3 \\ -5 & 8 \end{pmatrix}}}$$

$$3. \quad b^T c = b \cdot c = -1 \cdot 1 + 2 \cdot (-2) = -5 \neq 0$$

$\rightarrow b$ and c are not orthogonal.

Question 2

A group of biologists have observed lions in the jungle. They concluded that the number of individuals $N(t)$ at time $t > 0$ is given by the differential equation

$$\frac{dN(t)}{dt} = \frac{N(t)}{1000}(500 - N(t)).$$

1. Find $N(t)$ given that the population is $N(0) = 50$ at time $t = 0$.
2. What does the differential equation say about the population's per capita growth rate? What is the carrying capacity of the population?
3. Find the equilibrium points of the population growth model described above and classify them according to their stability.
4. Some biologists from the group that studied the population of lions have suggested that the growth model described above is not realistic for the following reason: it does not take into account the fact that when the number $N(t)$ of lions is very small, the population will decline because of the lack of mates for reproduction. Can you suggest a variant of the given differential equation that takes into account the fact that when the population of lions is less than 40 then the population will decrease to 0?

Question 3

The growth of a population of mice (krattspissmus) is described by the following Leslie matrix

$$L = \begin{pmatrix} 0 & 5 \\ 0.8 & 0 \end{pmatrix}.$$

1. What information does the matrix L provide about the population? (age groups, offspring rate, survival rates)
2. Find the eigenvectors and eigenvalues of L .
3. What is the biological interpretation of the larger eigenvalue of L ?
4. What is the percentage of each age group in the population after a large number of reproduction sessions? Justify your answer.

Question 4

Consider the function

$$f(x_1, x_2) = x_1^2 - 2x_1x_2 + x_2^3$$

defined on the closed and bounded domain

$$D = \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_1 \leq 2, -1 \leq x_2 \leq 2\}.$$

1. Find the global maximum and minimum of f on the domain D .
2. Does f have a local maximum or minimum at the point $(0, 0)$?
Does it have a local maximum or minimum at $(\frac{3}{2}, \frac{3}{2})$?
3. Find the equation of the tangent plane of the graph of f at the point $(1, 1)$.
4. Find the directional derivative of f at the point $(1, 1)$ towards the direction of the vector $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

