Solving a System of DEs - Example

Elisabeth Köbis, elisabeth.kobis@ntnu.no

Example

Solve

$$\frac{dy}{dt} = Ay,$$

where $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ and $A = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$ with initial condition $y(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

Solution: At first, we find the eigenvalues of A:

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda - 2 & 3 \\ -1 & \lambda + 2 \end{vmatrix} \\ &= (\lambda - 2)(\lambda + 2) + 3 \\ &= \lambda^2 - 4 + 3 = \lambda^2 - 1 \\ &= (\lambda - 1)(\lambda + 1). \end{aligned}$$

So, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$.

Example (cont.)

Now we find an eigenvector corresponding to $\lambda_1 = 1$:

$$Au = u \quad \Longrightarrow \quad \begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Longrightarrow \quad u_1 - 3u_2 = 0.$$

We can select any vector $u = (u_1, u_2)^T$ that fulfills the equation $u_1 - 3u_2 = 0$. Let us choose $u = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Example (cont.)

Now we find an eigenvector corresponding to $\lambda_2 = -1$:

$$Av = -v \quad \Longrightarrow \quad \begin{pmatrix} 3 & -3 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Longrightarrow \quad v_1 - v_2 = 0.$$

Let us choose $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The general solution is

$$y(t) = c_1 e^t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Example (cont.)

To find the unique solution, we use the initial condition $y(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and insert the initial values in the general solution to obtain the values for c_1, c_2:

$$\begin{aligned} y(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix} &= c_1 e^0 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Thus, we get the system

$$\begin{aligned} 3 &= 3c_1 + c_2 \\ -1 &= c_1 + c_2. \end{aligned}$$

The first equation yields $c_2 = 3 - 3c_1$. Plugging this into the second equation gives

$$-1 = c_1 + (3 - 3c_1) = -2c_1 + 3,$$

hence, $-4 = -2c_1$ and thus $c_1 = 2$. This gives $c_2 = 3 - 3c_1 = 3 - 3 \cdot 2 = -3$.

Example (cont.)

Therefore, the unique solution is

$$y(t) = 2e^t \begin{pmatrix} 3 \\ 1 \end{pmatrix} - 3e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$$